Lecture 7: Linear programming models

Duality and optimality

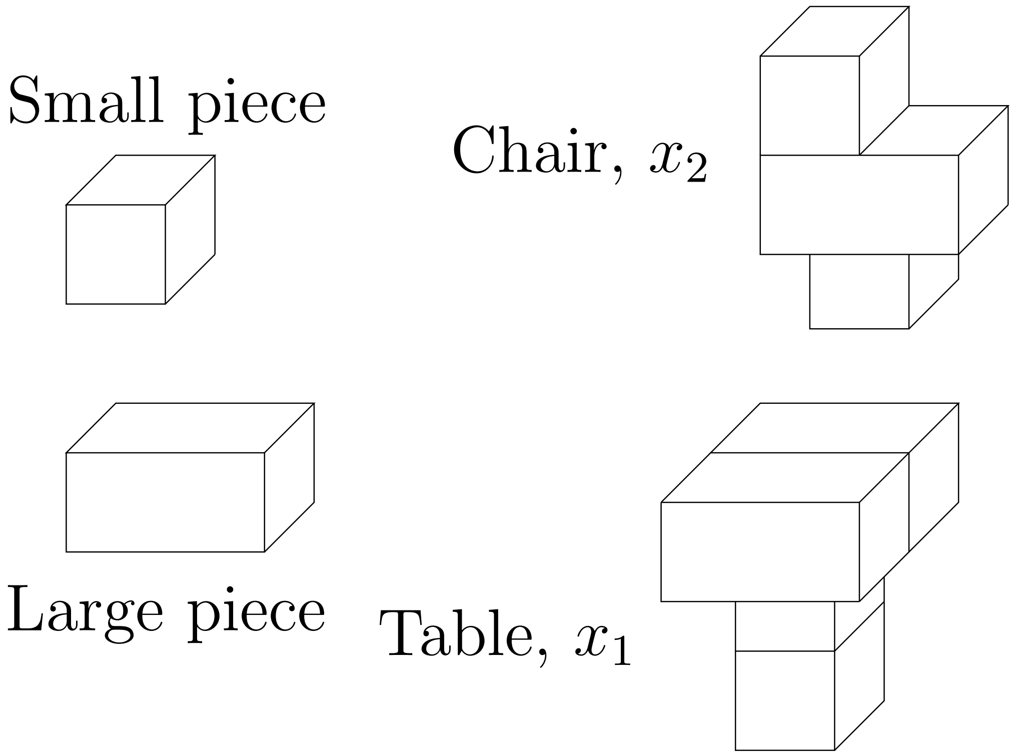
- LP: linear objective, linear constraints
- LP problems can be given a "standard form"
- LP problems are convex problems with a CQ fulfilled (linear constraints $\Longrightarrow$ Abadie)
- Strong duality holds; Lagrangian dual same as LP dual
- KKT necessary and sufficient!
- Simplex method: always satisfies complementarity; always primal feasibility after finding the first feasible solution; searches for a dual feasible point

Basic method and its foundations

- Know that if there exists an optimal solution, at least one of them is an extreme point (Thm. 8.10)
- Search only among extreme points
- Extreme points can be described in algebraic terms (Thm. 3.17). Find such a point
- Generate a descent direction; line search leads to the boundary! Choose direction so that the boundary point is an extreme point
- $\Longrightarrow$ Move to a neighbouring extreme point such that the objective value improves—the Simplex method!
- Convergence *finite*

An introductory problem—A DUPLO game

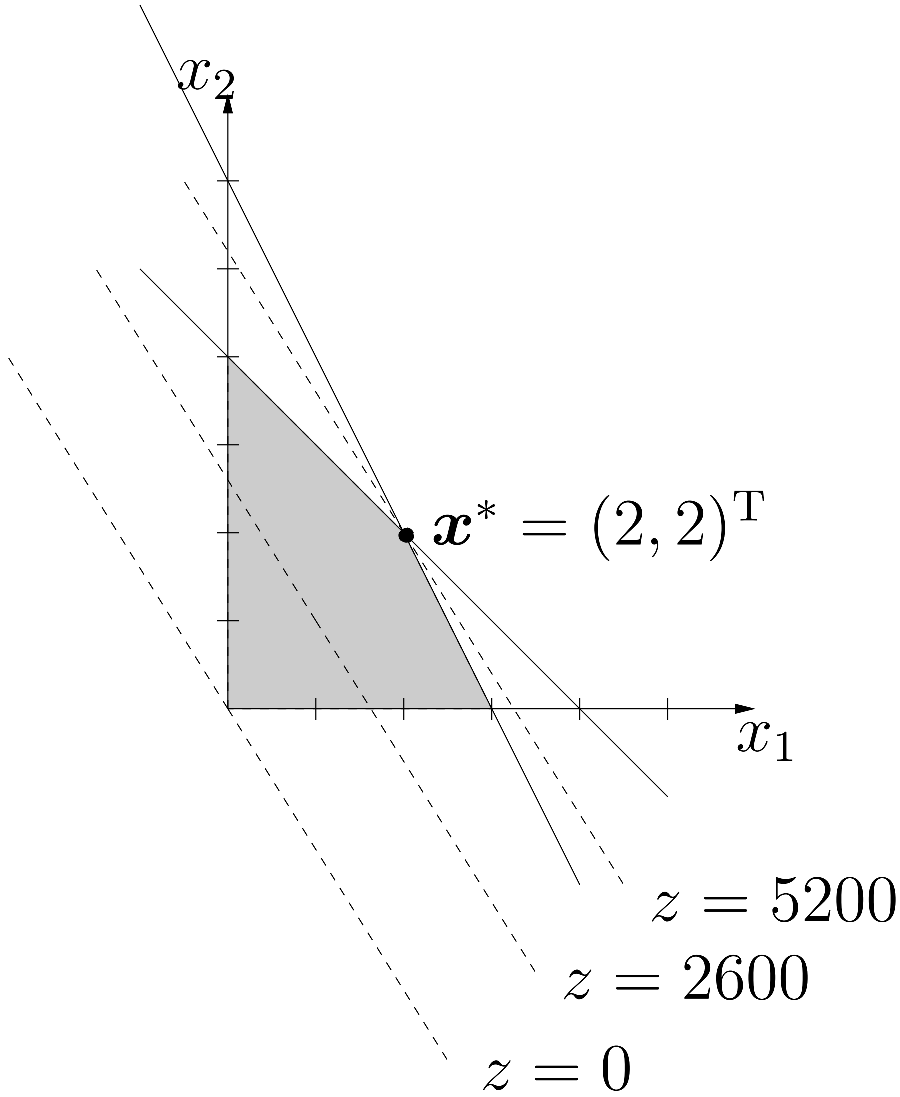
- A manufacturer produces two pieces of furniture: tables and chairs
- The production of furniture requires two different pieces of raw-material, large and small pieces
- One table is assembled from two pieces of each; one chair is assembled from one of the larger pieces and two of the smaller pieces



- Data: 6 large and 8 small pieces available. Selling a table gives 1600 SEK, a chair 1000 SEK
- Not trivial to choose an optimal production plan

- What is the problem and how do we solve it?
- Solution by (1) the DUPLO game; (2) graphically; (3) the Simplex method

$$\begin{array}{lrrrl} \text{maximize} & z = & 1600x_1 & +1000x_2 & \\ \text{subject to} & & 2x_1 & +x_2 & \leq 6, \\ & & 2x_1 & +2x_2 & \leq 8, \\ & & x_1, & x_2 & \geq 0 \end{array}$$



x_2
x_1
$\boldsymbol{x}^* = (2,2)^{\mathrm{T}}$
$z = 5200$
$z = 2600$
$z = 0$

Further topics

- Sensitivity analysis: What happens with z^*, $\boldsymbol{x}^*$ if $\ldots$?
- A dual problem: A manufacturer (Billy) produce book shelves with same raw material. Billy wish to expand their production; interested in acquiring our resources
- Two questions (with identical answers): (1) what is the lowest bid (price) for the total capacity at which we are willing to sell?; (2) what is the highest bid (price) that Billy are prepared to offer for the resources? The answer is a measure of the wealth of the company in terms of their resources (the shadow price)

A dual problem

- To study the problem, we introduce the variables

$$y_1 = \text{the price which Billy offers for each large piece,}$$
$$y_2 = \text{the price which Billy offers for each small piece,}$$
$$w = \text{the total bid which Billy offers}$$

- Example: Net income for a table is 1600 SEK; need to get at least price bid $\boldsymbol{y}$ such that $2y_1 + 2y_2 \geq 1600$

$$
\begin{array}{lrl}
\text{minimize} & w = & 6y_1 + 8y_2 \\
\text{subject to} & & 2y_1 + 2y_2 \geq 1600, \\
 & & y_1 + 2y_2 \geq 1000, \\
 & & y_1, \quad y_2 \geq 0
\end{array}
$$

- Why the sign? $\boldsymbol{y}$ is a price!
- Optimal solution: $\boldsymbol{y}^* = (600, 200)^{\mathrm{T}}$. The bid is $w^* = 5200$ SEK
- Remarks: (1) $z^* = w^*$! (Strong duality!) Our total income is the same as the value of our resources. (2) The price for a large piece equals its *shadow price*!

Geometric $\Longleftrightarrow$ Algebraic connections

- Must have equality constraints. Why? Inequalities cannot be manipulated while keeping the same solution set! Equalities can!
- Counter-example:

$$P = \{ \boldsymbol{x} \in \mathbb{R}^2_+ \mid x_1 + x_2 \geq 1; \quad 2x_1 + x_2 \leq 2 \}$$

- Good to know: Every polyhedron P can be described in the form

$$P = \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}; \quad \boldsymbol{x} \geq \mathbf{0}^n \}$$

- We call this the *standard form*

- Slack variables: ($\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$)

$$
\begin{aligned} \boldsymbol{A}\boldsymbol{x} &\le \boldsymbol{b}, \\ \boldsymbol{x} &\ge \boldsymbol{0}^n \end{aligned}
\quad\Longleftrightarrow\quad
\begin{aligned} \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}^m \boldsymbol{s} &= \boldsymbol{b}, \\ \boldsymbol{x} &\ge \boldsymbol{0}^n, \\ \boldsymbol{s} &\ge \boldsymbol{0}^m \end{aligned}
\quad\Longleftrightarrow\quad
\begin{aligned} [\boldsymbol{A}\,\boldsymbol{I}^m]\boldsymbol{v} &= \boldsymbol{b}, \\ \boldsymbol{v} &\ge \boldsymbol{0}^{n+m} \end{aligned}
$$

- We assume even that $\boldsymbol{b} \ge \boldsymbol{0}^m$; otherwise, multiply necessary rows by -1

- Idea: We describe an extreme point through this characterization of the feasible set; we then prove that moving between "adjacent" extreme points is simple

- *Basic feasible solution* (Algebra) $\Longleftrightarrow$ *Extreme point* (Geometry)
- Note: $\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{Ax} = \boldsymbol{b} \Longrightarrow$ Linear algebra
- $\boldsymbol{x} \geq \mathbf{0}^n : \boldsymbol{Ax} = \boldsymbol{b} \Longrightarrow$ Polyhedra, convex analysis
- Sign restrictions? If x_j is free of sign, substitute it everywhere by

  $$x_j = x_j^+ - x_j^-,$$

  where $x_j^+, x_j^- \geq 0$

DUPLO example with slack variables

$$
\begin{array}{llllllr}
\text{maximize}\; z = 1600x_1 + 1000x_2 & & & & & & \\
\text{subject to} & 2x_1 + & x_2 + & s_1 & & = 6 & (1) \\
 & 2x_1 + & 2x_2 & & + s_2 & = 8 & (2) \\
 & x_1, & x_2, & s_1, & s_2 & \geq 0 &
\end{array}
$$

Basic feasible solutions (BFS)

- Consider an LP in standard form:

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

  $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with rank $\boldsymbol{A} = m$ (otherwise, delete rows), $n > m$, and $\boldsymbol{b} \in \mathbb{R}^m_+$

- A point $\tilde{\boldsymbol{x}}$ is a *basic solution* if

  1. $\boldsymbol{A}\tilde{\boldsymbol{x}} = \boldsymbol{b}$; and
  2. the columns of $\boldsymbol{A}$ corresponding to the non-zero components of $\tilde{\boldsymbol{x}}$ are linearly independent

- A basic solution that satisfies non-negativity is called a *basic feasible solution* (BFS)
- Additional terms: degenerate, non-degenerate basic solutions
- Connection BFS–extreme points?
- *A point* $\boldsymbol{x}$ *is an extreme point of the set* $\{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b};\ \boldsymbol{x} \geq \mathbf{0}^n \,\}$ *if and only if it is a basic feasible solution*
- Proof by the fact that the rank of $\boldsymbol{A}$ is full + Thm. 3.17 (algebraic char. of extreme points)

The Representation Theorem revisited

Let $P = \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b};\ \boldsymbol{x} \geq \boldsymbol{0}^n \}$ *and* $V = \{\boldsymbol{v}^1, \dots, \boldsymbol{v}^k\}$ *its set of extreme points. If and only if* P *is nonempty,* V *is nonempty (finite). Let* $C = \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}^m;\ \boldsymbol{x} \geq \boldsymbol{0}^n \}$ *and* $D = \{\boldsymbol{d}^1, \dots, \boldsymbol{d}^r\}$ *be the set of extreme directions of* C*. If and only if* P *is unbounded* D *is nonempty (finite). Every* $\boldsymbol{x} \in P$ *is the sum of a convex combination of points in* V *and a non-negative linear combination of points in* D*:*

$$\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i \boldsymbol{v}^i + \sum_{j=1}^{r} \beta_j \boldsymbol{d}^j,$$

where $\alpha_1, \dots, \alpha_k \geq 0$*:* $\sum_{i=1}^{k} \alpha_i = 1$*, and* $\beta_1, \dots, \beta_r \geq 0$

Existence of optimal solutions to LP

- *Let the sets* P, V *and* D *be defined as in the above theorem and consider the LP*

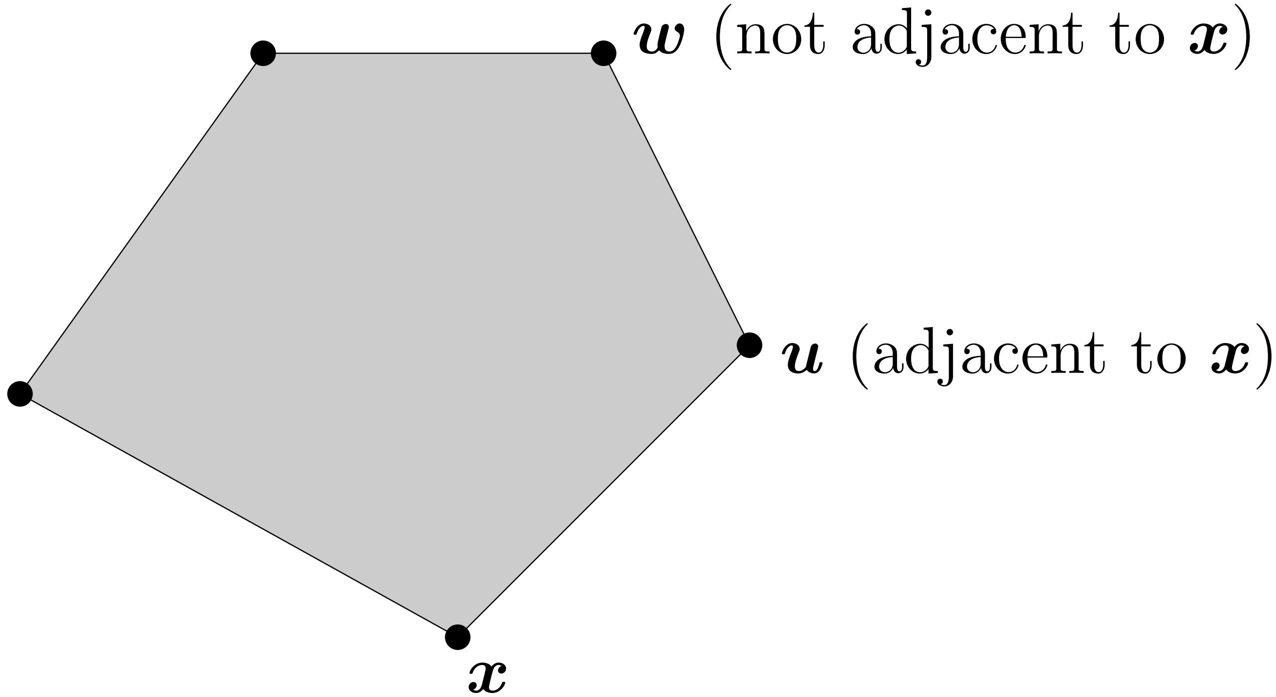
$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{x} \in P \end{aligned}$$

  This problem has a finite optimal solution if and only if P *is nonempty and* z *is lower bounded on* P, *that is, if* $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}^j \geq 0$ *for all* $\boldsymbol{d}^j \in D$. *If the problem has a finite optimal solution, then there exists an optimal solution among the extreme points*

- *Proof.*

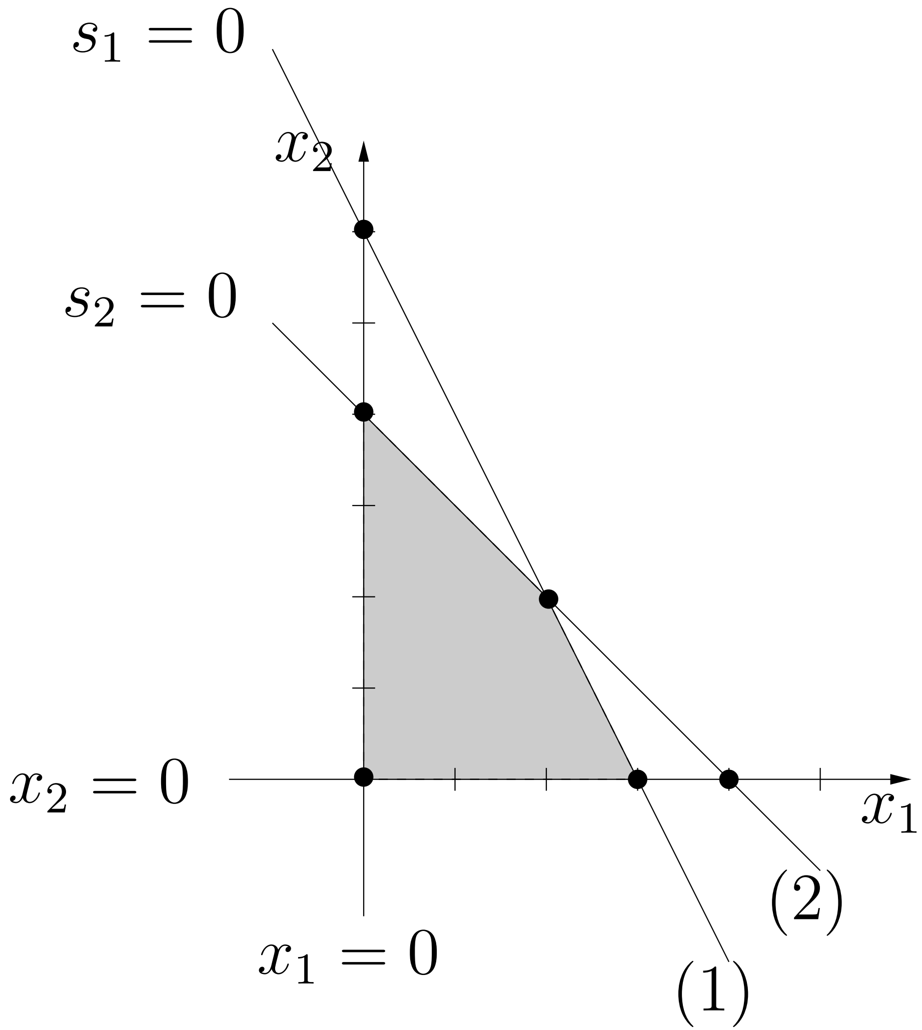
Adjacent extreme points

- Consider the following polytope:



- No point on the line segment joining $\boldsymbol{x}$ and $\boldsymbol{u}$ can be written as a convex combination of any pair of points that are not on this line segment. However, this is not true for the points on the line segment between the extreme points $\boldsymbol{x}$ and $\boldsymbol{w}$. The extreme points $\boldsymbol{x}$ and $\boldsymbol{u}$ are said to be *adjacent* (while $\boldsymbol{x}$ and $\boldsymbol{w}$ are not adjacent)

- *Two extreme points are adjacent if and only if there exist corresponding BFSs whose sets of basic variables differ in exactly* one *place*


$s_1 = 0$
x_2
$s_2 = 0$
$x_2 = 0$
x_1
(2)
$x_1 = 0$
(1)