Chalmers/Gothenburg University
Mathematical Sciences

**EXAM**

# TMA947/MAN280
# OPTIMIZATION, BASIC COURSE

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| **Date:** | 08–03–25 |
| **Time:** | House V, morning |
| **Aids:** | Text memory-less calculator, English–Swedish dictionary |
| **Number of questions:** | 7; passed on one question requires 2 points of 3.<br>Questions are *not* numbered by difficulty.<br>To pass requires 10 points and three passed questions. |
| **Examiner:** | Michael Patriksson |
| **Teacher on duty:** | Adam Wojciechowski (0762-721860) |
| **Result announced:** | 08–04–02<br>Short answers are also given at the end of<br>the exam on the notice board for optimization<br>in the MV building. |

## Exam instructions

### When you answer the questions

*Use generally valid theory and methods.*
*State your methodology carefully.*

*Only write on one page of each sheet. Do not use a red pen.*
*Do not answer more than one question per page.*

### At the end of the exam

*Sort your solutions by the order of the questions.*
*Mark on the cover the questions you have answered.*
*Count the number of sheets you hand in and fill in the number on the cover.*

## Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{minimize} \quad & z = -x_1 - x_2, \\
\text{subject to} \quad & -x_1 - 2x_2 - x_3 = 2, \\
& 3x_1 + x_2 \leq -1, \\
& x_2, \; x_3 \geq 0, \\
& x_1 \in \mathbb{R} \text{ (free)}.
\end{aligned}$$

**(2p)** a) Solve this problem by using phase I and phase II of the simplex method. [*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

**(1p)** b) Motivate using the solution from a) and the relationships between primal and dual problems why there cannot exist a vector $\boldsymbol{u} = (u_1, u_2, u_3)^{\mathrm{T}}$ fulfilling the following system of constraints:

$$\begin{pmatrix} -1 & 3 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix} \boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_1 \geq 0, \; u_2 \leq 0, \; u_3 \geq 0.$$

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## Question 2

(modelling)

Consider the mixed-integer problem (MIP) of minimizing the linear function $f(\boldsymbol{x}, \boldsymbol{y})$ over the set $X \times Y$, where $X = \{ \boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \}$ and $Y = \{ \boldsymbol{y} \in \mathbb{R}^m \mid y_i \geq 0, \; i = 1, \dots, m \}$.

**(1p)** a) Formulate the mixed-integer problem as *one* non-linear program using *only continuous* variables and continuous constraints.

**(2p)** b) Assume that $n = 1$. Explain how to solve the mixed-integer problem by solving *a number of* linear programs. Formulate these programs.

## Question 3

(topics in Lagrangian duality)

Consider the problem to find

$$
\begin{aligned}
f^* := \underset{x}{\text{infimum}} \;\; & f(\boldsymbol{x}), && (1)\\
\text{subject to} \;\; & g_i(\boldsymbol{x}) \leq 0, \qquad i = 1, \dots, m,\\
& \boldsymbol{x} \in X,
\end{aligned}
$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, m$, are given functions, and $X \subseteq \mathbb{R}^n$.

Consider also the Lagrangian dual problem to find

$$
q^* := \underset{\boldsymbol{\mu} \geq 0^m}{\text{supremum}} \;\; q(\boldsymbol{\mu}), \tag{2}
$$

where

$$
q(\boldsymbol{\mu}) = \underset{x \in X}{\text{infimum}} \;\; L(\boldsymbol{x}, \boldsymbol{\mu}),
$$

and the function $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$
L(\boldsymbol{x}, \boldsymbol{\mu}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \mu_i g_i(\boldsymbol{x}).
$$

**(1p)** a) Establish that the optimization problem (2) is a convex problem.

**(1p)** b) Suppose that all the functions $f$ and $g_i$, $i = 1, 2, \dots, m$, are continuous and that $X$ is nonempty, closed and bounded. Establish that the function $q$ is finite on $\mathbb{R}^m$.

**(1p)** c) Take as an example $f(x) := x$, $m = 1$ and $g_1(x) = \frac{1}{2}x^2$, and $X := \mathbb{R}$. What is the optimal primal solution (if any)? What is the optimal dual solution (if any)? Letting $\Gamma := f^* - q^*$ denote the "duality gap" of the problem, what is the value of $\Gamma$ in this instance?

## (3p) Question 4

(complementarity slackness theorem)

Consider the primal–dual pair of linear programs given by

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{c}, \\ & \boldsymbol{y} \geq \boldsymbol{0}^m. \end{aligned} \tag{2}$$

THEOREM 1 (Complementary Slackness Theorem) *Let $\boldsymbol{x}$ be a feasible solution to (1) and $\boldsymbol{y}$ a feasible solution to (2). Then $\boldsymbol{x}$ is optimal to (1) and $\boldsymbol{y}$ optimal to (2) if and only if*

$$x_j(c_j - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}_{\cdot j}) = 0, \qquad j = 1, \dots, n, \tag{3a}$$
$$y_i(\boldsymbol{A}_{i\cdot}\boldsymbol{x} - b_i) = 0, \qquad i = 1 \dots, m, \tag{3b}$$

*where $\boldsymbol{A}_{\cdot j}$ is the $j^{\text{th}}$ column of $\boldsymbol{A}$ and $\boldsymbol{A}_{i\cdot}$ the $i^{\text{th}}$ row of $\boldsymbol{A}$.* ■

Prove this theorem. If you wish to refer to other theorems from The Book in your proof, then state (but do not prove) those theorems, as they apply to the problem given.

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## Question 5

(quadratic programming)

**(1p)** a) Consider the quadratic problem:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{aligned} \tag{QP}$$

where $\boldsymbol{H} \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ has full row rank, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. Set up the KKT-conditions and find the optimal Lagrange multipliers.

**(1p)** b) The Lagrange dual problem to (QP) is also a quadratic problem. State the (quadratic) dual problem and show that the dual solution is identical to the Lagrange multipliers in problem a).

**(1p)** c) Let $\boldsymbol{Z} \in \mathbb{R}^{n\times(n-m)}$ be the null-space matrix to $\boldsymbol{A}$ in (QP), i.e., $\boldsymbol{AZ} = \boldsymbol{0}^{m\times(n-m)}$. Assume $\boldsymbol{H}$ is neither positive definite nor positive semidefinite, but that $\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{HZ}$ is positive semidefinite. Is a local optimal solution in (QP) a global optimal solution? Answer true or false, and motivate your answer!

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## (3p) Question 6

(the Frank-Wolfe algorithm)

Consider the problem

$$\begin{aligned}
\text{minimize} \quad & \tfrac{1}{2}x_1^2 - \tfrac{1}{2}(x_2-1)^2, \\
\text{subject to} \quad & x_1 \leq 2, \\
& 0 \leq x_2 \leq 2, \\
& 1-4x_1 \leq x_2 \leq 1+4x_1.
\end{aligned}$$

Start at $\boldsymbol{x}^0 = (1,1)^{\mathrm{T}}$ and perform *one*(!) complete iteration with the Frank-Wolfe algorithm. Is the resulting vector $\boldsymbol{x}^1$ a KKT-point? Is it a local minimum? Is it a global minimum? Motivate your answers!

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## Question 7

(nonlinear optimization solves interesting problems)

**(1p)** a) Fermat's Last Theorem states that there are no solutions in the positive integers of the equation

$$x^n + y^n = z^n,$$

for $n \geq 3$. Re-state this problem as a continuous nonlinear program, whose optimal solution reveals the answer to the above question.

**(1p)** b) Show that for any symmetric and positive definite matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ there exists a positive number $c$ such that

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax} \geq c\|\boldsymbol{x}\|^2, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

**(1p)** c) An $n \times n$ matrix $\boldsymbol{A}$ is said to be *invertible* if there exists for each vector $\boldsymbol{y} \in \mathbb{R}^n$ a unique vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{Ax} = \boldsymbol{y}$. (Then, there is a unique $n \times n$ matrix $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}^{-1}\boldsymbol{y} = \boldsymbol{x}$ precisely if $\boldsymbol{Ax} = \boldsymbol{y}$. The matrix $\boldsymbol{A}^{-1}$ is then denoted the *inverse* of $\boldsymbol{A}$.)

Show that if $\boldsymbol{A}$ is a positive definite and symmetric $n \times n$ matrix then $\boldsymbol{A}$ is invertible.

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*Good luck!*