Ex 1

$$\min \; f(x) = x_1 + 2x_2^2 + 3x_3^3$$

subj. to

$$x_1 + 2x_2 + x_3 \leq 3 \qquad (1)$$

$$2x_1^2 + x_2 \geq 2 \qquad (2)$$

$$2x_1 + x_3 = 2 \qquad (3)$$

$$x_1, x_2, x_3 \geq 0$$

a) Formulate Lagr. dual problem that origin. from relaxation of (1) - (3)

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$$\min \; f(x)$$

subj. to

$$g_1(x) = x_1 + 2x_2 + x_3 - 3 \leq 0$$

$$g_2(x) = -2x_1^2 - x_2 + 2 \leq 0$$

$$h(x) = 2x_1 + x_3 - 2 = 0$$

$$x \in X = \{ x_1, x_2, x_3 \geq 0 \}$$

$$L(x,\mu,\lambda) = f(x) + \mu_1 g_1(x) + \mu_2 g_2(x) + \lambda h(x)$$

$$q(\mu,\lambda) = \min_{x \in X} L(x,\mu,\lambda)$$

$$q(\mu,\lambda) = \min_{x\in X} \; x_1 + 2x_2^2 + 3x_3^3 + \mu_1(x_1+2x_2-3)$$

$$+\mu_2(-2x_1^2 - x_2 + 2) + \lambda(2x_1 + x_3 - 2)$$

$$= \min_{x_1\geq 0} \underbrace{(1+\mu_1+2\lambda)x_1 - 2\mu_2 x_1^2}_{\varphi_1(x_1)}$$

$$+ \min_{x_2\geq 0} \underbrace{(2\mu_1-\mu_2)x_2 + 2x_2^2}_{\varphi_2(x_2)}$$

$$+ \min_{x_3\geq 0} \underbrace{\lambda x_3 + 3x_3^3}_{\varphi_3(x_3)} \quad + 2\mu_2 - 3\mu_1 - 2\lambda$$

where $\mu \geq 0$ $\lambda\in\mathbb{R}$

$$\varphi_1'(x_1) = 1+\mu_1+2\lambda - 4\mu_2 x_1 \qquad \varphi''(x_1) = -4\mu_2$$

$\therefore$ $\varphi_1$ concave $\qquad \varphi_1'(0) = \cancel{1+\mu_1+2\lambda} = 0$

$$\lim_{x_1\to\infty} \varphi_1(x_1) = \begin{cases} -\infty & \mu_2 > 0 \\ & \text{or } \lambda < \frac{-1-\mu_1}{2} \\ 0 & \mu_2 = 0 \quad 1+\mu_1+2\lambda = 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$\min_{x_1\geq 0} \varphi_1(x_1) = \begin{cases} -\infty & \mu_2 > 0 \text{ or } \\ & \lambda < -\frac{1-\mu_1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_2'(x_2) = 4x_2 + 2\mu_1 - \mu_2 \qquad \varphi_2''(x_2) = 4 > 0$$

$$\varphi_2'(x_2) = 0 \iff x_2 = \frac{\mu_2 - 2\mu_1}{4}$$

$$\varphi_2\left(\frac{\mu_2 - 2\mu_1}{4}\right) = 2\left(\frac{\mu_2 - 2\mu_1}{4}\right)^2 + (2\mu_1 - \mu_2)\left(\frac{\mu_2 - 2\mu_1}{4}\right)$$

$$= \frac{1}{8}(\mu_2 - 2\mu_1)^2 - \frac{2}{8}(\mu_2 - 2\mu_1)^2$$

$$= -\frac{1}{8}(\mu_2 - 2\mu_1)^2$$

$$\varphi_3'(x_3) = 9x_3^2 + (\mu_1 + \lambda) \qquad \varphi_3''(x_3) = 18x_3 \geq 0$$

$$\varphi_3'(x_3) = 0 \iff x_3 = \sqrt{\frac{-(\mu_1 + \lambda)}{3}} \qquad \mu_1 + \lambda \leq 0$$

$$\varphi_3(0) = 0 \qquad \varphi_3\left(\sqrt{\frac{|\mu_1 + \lambda|}{3}}\right) = \frac{3\,|\mu_1 + \lambda|^{3/2}}{3^3} - |\mu_1 + \lambda|^{3/2}$$

$$\lim_{x_3 \to \infty} \varphi_3(x_3) = \infty \qquad = -\frac{2}{3}|\mu_1 + \lambda|^{3/2}$$

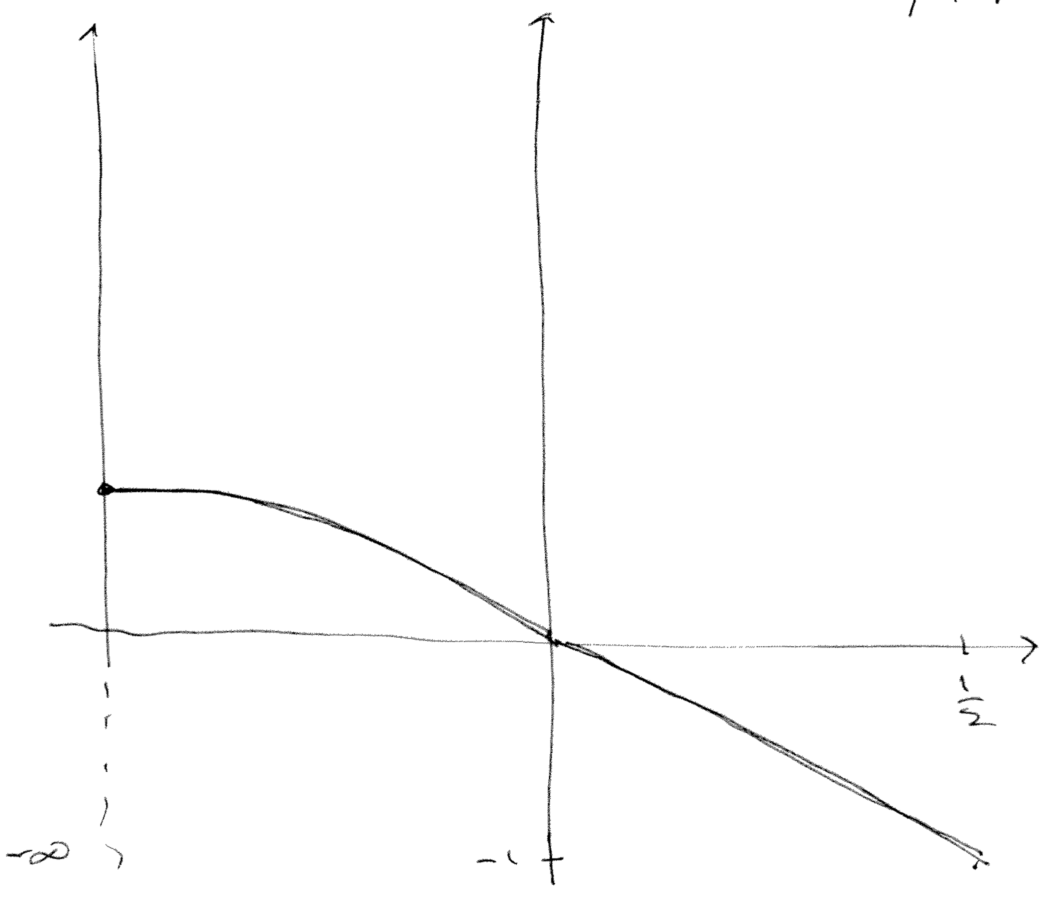
$$\min_{x_3 \geq 0} \varphi_3(x_3) = \begin{cases} -\frac{2}{3}|\mu_1 + \lambda|^{3/2} & \mu_1 + \lambda \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$
q(\lambda,\mu) = \begin{cases} -\infty & \mu_2 > 0 \ \text{ or } \ \mu_1 < -2\lambda - 1 \\ -\frac{4}{8}\mu_1^2 - \frac{2}{3}|\mu_1+\lambda|^{3/2} - 3\mu_1 + 2\overset{0}{\mu_2} - 2\lambda & \mu_2 = 0 \quad -\lambda \geq \mu_1 \geq -2\lambda - 1 \\ -\frac{4}{8}\mu_1^2 - 3\mu_1 + 2\overset{0}{\mu_2} - 2\lambda & \mu_2 = 0 \quad \mu_1 > -\lambda \end{cases}
$$

$\mu_1 = 0 \quad \mu_2 = 0$

$$
q(0,0,\lambda) = \begin{cases} -\frac{2}{3}|\lambda|^{3/2} - 2\lambda & -\frac{1}{2} \leq \lambda \leq 0 \\ -2\lambda & \lambda \geq 0 \end{cases}
$$

$\mu_1 = \mu_2 = 0$



Ex 2  min $z = c^T x$

$$Ax \geq b \quad (1)$$
$$x \geq 0$$

~~Formulera~~ .

Formulate the dual by relaxing (1).

dual: $\max_{\mu \geq 0} q(\mu)$  $X = \{x : x \geq 0\}$

~~$q(\mu) = \min_{x \in X} \{$~~

linear func. over polyhedron

$$q(\mu) := \inf_{x \in X} L(x,\mu) = \min_{x \in X} \{ c^T x + \mu^T (b - Ax) \}$$

$$= \min_{x \in X} \{ (c^T - \mu^T A) x \} + \mu^T b$$

$$= \begin{cases} \mu^T b & c^T - \mu^T A \geq 0 \\ -\infty & (c^T - \mu^T A)_i < 0 \quad \text{some } i \in \{1,\dots,n\} \end{cases}$$

The dual becomes:
$$\max \ \mu^T b$$
$$c^T - \mu^T A \geq 0$$
$$\mu \geq 0$$

$$\iff \max \ b^T \mu$$
$$A^T \mu \leq c$$
$$\mu \geq 0$$

Ex 3 $\quad \min f(x) = x_1^2 + 2x_2^2$ (Prim-Dual opt. cond.)

subj. to $\quad x_1 + x_2 \geq 2,$

$\quad x_1^2 + x_2^2 \leq 5$

Find opt. sol.

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$\min f(x)$

$g_1(x) = -x_1 - x_2 + 2 \leq 0$

$g_2(x) = x_1^2 + x_2^2 - 5 \leq 0$

1) Formulate lagr. function & dual function

$$L(x,\mu) = f(x) + \mu_1 g_1(x) + \mu_2 g_2(x)$$

$$q(\mu) = \min_{x \in \mathbb{R}^2} L(x,\mu) = \min_{x \in \mathbb{R}^2} x_1^2 + 2x_2^2 + \mu_1(-x_1 - x_2 + 2)$$

$$+ \mu_2(x_1^2 + x_2^2 - 5) =$$

$$= \min_{x \in \mathbb{R}^2} x_1^2(1+\mu_2) - \mu_1 x_1 + x_2^2(2+\mu_2) - \mu_1 x_2 + 2\mu_1 - 5\mu_2$$

$$= \min_{x_1 \in \mathbb{R}} \underbrace{x_1^2(1+\mu_2) - \mu_1 x_1}_{\varphi_1(x)} + \min_{x_2 \in \mathbb{R}} \underbrace{x_2^2(2+\mu_2) - \mu_1 x_2}_{\varphi_2(x)}$$

$$+ 2\mu_1 - 5\mu_2$$

$\varphi'(x_1) = 2(1+\mu_2)x_1 - \mu_1 \qquad \varphi''(x_2) = 2(1+\mu_2) > 0$

$\varphi'(x_1) = 0 \iff x_1 = \dfrac{\mu_1}{2(1+\mu_2)} \geq 0$

(since we have a strictly convex problem, this is the unique global min.)

same way $\varphi_2$ gives $x_2 = \dfrac{\mu_1}{2(2+\mu_2)}$

choose $\mu \geq 0$ 1) fulfilled

let $x = \left( \dfrac{\mu_1}{2(1+\mu_2)}, \dfrac{\mu_1}{2(2+\mu_2)} \right)$ and 2) fulfilled

4.) gives $\mu_i g_i(x) = 0$

$$\Rightarrow \mu_1(-x_1 - x_2 + 2) = \mu_1\left( \frac{-\mu_1}{2(1+\mu_2)} - \frac{\mu_1}{2(2+\mu_2)} + 2 \right) = 0 \qquad (i)$$

$$\mu_2(x_1^2 + x_2^2 - 5) = \mu_2\left( \frac{\mu_1^2}{4(1+\mu_2)^2} - \frac{\mu_1^2}{4(2+\mu_2)^2} - 5 \right) = 0 \qquad (ii)$$

if $\mu_1 = 0$ $g_1(x(\mu)) = 2 > 0$ 3) not sat.

$\Rightarrow \mu_1 \neq 0$

if $\mu_2 = 0 \overset{(i)}{\Rightarrow} \dfrac{\mu_1}{2} + \dfrac{\mu_1}{4} = 2 \iff \mu_1 = \dfrac{8}{3}$

$\mu^* = \left(\frac{8}{3}, 0\right) \qquad x^* = \begin{pmatrix} 4/3 \\ 2/3 \end{pmatrix}$

$\mu^* \geq 0 \qquad g_1(x^*) = -\frac{4}{3} - \frac{2}{3} + 2 = 0$

$g_2(x^*) = \frac{16+4}{9} - 5 = \frac{20-45}{9} < 0$

1) - 4) satisfied

$\Rightarrow$ $x^*$ opt. solution

$f(x^*) = \ldots = \frac{8}{3}$ opt. obj. func. value.

<u>Ex 4</u> $\quad \min f(x) = \frac{1}{2}\|y-x\|^2$

subj. to $\quad Ax = 0^m$

$A$ rank $m$

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Convex problem + affine constr

<u>Thm 6.11</u> $\Rightarrow \exists$ lagr. mult. $\lambda^*$ $\quad q(\lambda^*) = f^*$ (no duality gap)

$\uparrow$ in eq. constr. use $\lambda$ as mult.

$$L(x,\lambda) = \frac{1}{2}\|y-x\|^2 + \lambda^T A x$$

wish to find $q(\lambda) = \min_{x \in X} L(x,\lambda)$ $\qquad X = \mathbb{R}^n$

$$\nabla_x L(x,\lambda) = x - y + A^T\lambda$$

$\nabla_x^2 L(x,\lambda) = I$ pos. def. $\Rightarrow$ convex prob. $\Rightarrow \nabla_x f = 0$ suff.

$\cancel{\nabla} \Rightarrow x = y - A^T\lambda$

$$q(\lambda) = \frac{1}{2}\|A^T\lambda\|^2 + \lambda^T A(y - A^T\lambda)$$

$$= \frac{1}{2}\|A^T\lambda\|^2 + \lambda^T A y - \lambda^T A A^T \lambda$$

$$= \frac{1}{2}(A^T\lambda)^T A^T\lambda + \lambda^T A y - \lambda^T A A^T \lambda =$$

$$= \lambda^T A y - \frac{1}{2}\lambda^T A \lambda$$

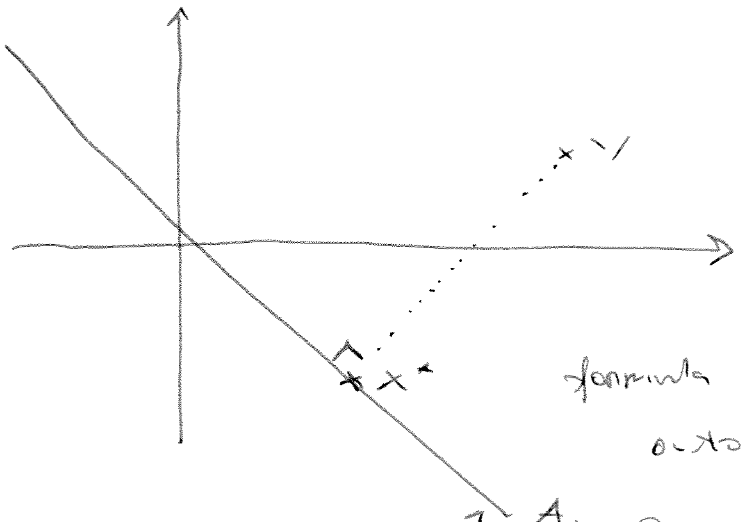
$q$ diffbl. + concave
$\Rightarrow \nabla_\lambda q = 0$ suff. opt. cond.

$$\nabla_\lambda q(\lambda) = Ay - AA^T\lambda = 0$$

$$\Leftrightarrow \quad \lambda = (AA^T)^{-1} A y$$

exists for rank $m$ matrices

$$x^* = y - A^T\lambda = y - A^T(AA^T)^{-1}Ay = \left(I - A^T(AA^T)^{-1}A\right)y$$




formula for proj. $y$ onto $Ax = 0$.

$Ax = 0$

subsp. incl. origin

Ex 5 $\min f(x) = -x_1 + x_2$

subj. to $x_1^2 + x_2^2 \leq 25$

$x_1 - x_2 \leq 1$

Is $x^* = (4 \quad 3)^T$ a glob. min

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Check if Prim-Dual opt. cond. are fulfilled

$\min f(x)$

$g_1(x) = x_1^2 + x_2^2 - 25 \leq 0$

$g_2(x) = x_1 - x_2 - 1 \leq 0$

Form. lagr. func:

$L(x,\mu) = f(x) + \mu_1 g_1(x) + \mu_2 g_2(x)$

$= -x_1 + x_2 + \mu_1(x_1^2 + x_2^2 - 25) + \mu_2(x_1 - x_2 - 1)$

$= \underbrace{\mu_1 x_1^2 + (\mu_2 - 1)x_1}_{\varphi_1(x_1)} + \underbrace{\mu_1 x_2^2 + (1-\mu_2)x_2}_{\varphi_2(x_2)}$

$-25\mu_1 - \mu_2$

$\min_{x \in X} L(x,\mu) \qquad X = \mathbb{R}^2$

$\varphi_1'(x_1) = 2\mu_1 x_1 + \mu_2 - 1$ $\qquad$ $\varphi_1''(x_1) = 2\mu_1 \geq 0$ conv.

$\varphi_2(x_2) = 2\mu_1 x_2 + 1 - \mu_2$ $\qquad$ $\varphi_2''(x_2) = 2\mu_1 \geq 0$ conv.

$\Rightarrow$ $\varphi_1'(x_1) = 0$ suff. $\Rightarrow$ $\cancel{x_1 = \frac{1-\mu_2}{2\mu_1}}$ $\qquad$ $2\mu_1 x_1 + \mu_2 = 1$

$\varphi_2'(x_2) = 0$ suff. $\Rightarrow$ $\cancel{x_2 = \frac{\mu_2 - 1}{2\mu_1}}$ $\qquad$ $2\mu_1 x_2 - \mu_2 = -1$

$x = (4 \quad 3)$ gives

$$\begin{cases} 8\mu_1 + \mu_2 = 1 \\ 6\mu_1 - \mu_2 = -1 \end{cases}$$

$\Rightarrow$ $14\mu_1 = 0$ $\quad$ $\mu_1 = 0$ $\qquad$ $\mu_2 = 1$

1) ok $\qquad$ 2) ok $\qquad$ 3) ok if x feas:

$g_1(x) = 4^2 + 3^2 - 25 = 0$

$g_2(x) = 4 - 3 - 1 = 0$

4) ok

$\therefore$ $(\mu, x)$ lagr. mult + opt sol

$\Rightarrow x$ opt. sol

<u>Comment:</u> If $\nexists \mu$ s.t. $(\mu, x)$ sat. 1)–4) we generally do not know that $x$ non-opt.
If problem is convex + inner point thm 6.10 says 1)–4) necessary