TMA 965 : Discrete Matematics D3-fall, 03

Homework1

Warm up problems to keep you busy for the first week and to solidify your background knowledge in combinatorics

1. A group of 8 right-handed and 6 left-handed ping pong players are in a room. In how many ways can they pair off so that only one pair consists of two right-handed players?

2. How many path of minimal length, along the edges, are there from the origin to the point $(1, 1, ..., 1)$ in the unit cube?

3. How many five-digit telephone numbers have a digit which occurs more than once?

4. A family consists of 9, people, 5 men and 4 woman. They are going to play a game where one first must choose 3 referees, 2 who are men and 1 who is a woman. Then the remaining 6 people have to be paired off so that each pair consists of a woman and a man. In how many ways can this be done?

5. In the soccer world championships, there are 32 countries. The 32 teams will be divided into 8 groups, each group having 4 teams. In how many ways, can this be done if the internal order within each group doesn't matter but the order of the groups themselves does matter.

6. The world's best soccer club, Liverpool FC (!!!), last year had a team consisting of 22 players : 3 goal keepers, 6 defenders, 9 midfield players and

4 offensive players. The trainer prefers the 4-4-2 system which means that there is 1 goal keeper, 4 defenders, 4 midfield players and 2 offensive players. How many different combinations can play a game if (a) the internal order within each category of player matters; (a) the internal order within each category of player does not matter.

A few more perhaps more challenging problems: For some of these, it might help to see some more material presented first but it doesn't hurt to start thinking about them right away.

1. When the mathematician Gauβ was 9 years old, he was so bored in school that one day he proved the following well-known formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

In today's schools, one usually proves this by induction; do this. But Gauβ had another much simpler proof; can you find it?

2. Let S be a finite set. Prove *combinatorially* that the number of subsets of S that have an even number of elements is equal to the number of subsets of S that have an odd number of elements. One can also obtain this as a consequence of the binomial formula but the point is to do this *combinatorially* which means to describe an explicit 1 to 1 correspondence between the two collections.

3. Suppose we have a number of different subsets of $\{1, 2, \ldots, 8\}$, with the property that each one has four members, and each member of $\{1, 2, \ldots, 8\}$ belongs to exactly three of the subsets. How many subsets are there? Write down a collection of subsets which satisfies the conditions.

4. Let $(n)_m = n(n-1)\ldots,(n-m+1)$. By interpreting the result in terms of ordered selections, show that

$$(n)_m(n-m)_{r-m} = (n)_r$$

for any positive integers satisfying $n > r > m$.

5. Let q_n be the number of words of length n with alphabet $\{a, b, c, d, e, f\}$ that has an odd number of b:s. Prove that

$$q_{n+1} = 6^n + 4q_n.$$

(Hint : Divide up the words of length $n+1$ according to whether they begin with a b or not.).

6. Prove the following identity combinatorially.

$$\binom{s-1}{0} + \binom{s}{1} + \ldots + \binom{s+n-2}{n-1} + \binom{s+n-1}{n} = \binom{s+n}{n}$$

7. An ordered triple (a, b, c) av whole integers is called an *arithmetic progression (AP)* if $b - a = c - b$. Give a formula for the number of AP's that consist of numbers from $\{1, 2, ..., n\}$.

8. Give and prove a simpler expression for

$$\sum_{k=0}^{n} k \cdot \binom{n}{k}$$

OBSERVE! You get more points if your proof does not use the formula for $\binom{n}{k}$; this would mean you give a combinatorial proof.