TMA970 Inledande matematisk analys F/TM

Lösningar 29/8 -14

1. (a) divergent; (b) konvergent;
(c) divergent; (d) konvergent;
(e) falskt; (f) sant;
(g) falskt; (h) falskt.

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2. (a)
$$\left(\frac{x+3}{x-1}\right)^{2x+1} = \left(\left(1+\frac{4}{x-1}\right)^{\frac{x-1}{4}}\right)^{4\cdot\frac{2x+1}{x-1}} =$$
$$= \Bigg(\underbrace{\left(1+\frac{4}{x-1}\right)^{\frac{x-1}{4}}}_{\to\, e,\ x\to\infty}\Bigg)^{\left(8+\underbrace{\frac{12}{x-1}}_{\to 0}\right)} \longrightarrow e^{8}$$

(b)
$$\sqrt{x^2-2x-1} - \sqrt{x^2-7x+3} = (\text{"}\infty-\infty\text{"})$$
$$= \frac{(\cancel{x^2}-2x-1)-(\cancel{x^2}-7x+3)}{\sqrt{x^2-2x-1}+\sqrt{x^2-7x+3}} = \frac{5x-4}{\underbrace{|x|}_{=-x}\left(\sqrt{1-\frac{2}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{7}{x}+\frac{3}{x^2}}\right)} =$$
$$= -\frac{5-\frac{4}{x}}{\sqrt{1-\frac{2}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{7}{x}+\frac{3}{x^2}}} \xrightarrow[x\to-\infty]{} -\frac{5}{2}$$

$x\to-\infty$
$\Rightarrow x<0$

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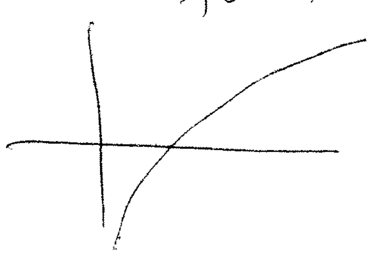
3. $\boxed{D_{f_\alpha}:\ x>0}$ oavsett $\alpha\in\mathbb{R}$

($\alpha=0$: $f_0(x)=\ln x$



känd funktion)

$\alpha > 0$: $f_\alpha(1) = 0$, enda nollstället

$f_\alpha(x) < 0$ för $0 < x < 1$, $f_\alpha(x) > 0$ för $x > 1$.

$\lim_{x \to 0_+} f_\alpha(x) = -\infty$

$\Rightarrow$ $x = 0$ vertikal asymptot

$\lim_{x \to +\infty} f_\alpha(x) = 0 \Rightarrow y = 0$ horisontell asymptot

$$f_\alpha'(x) = \frac{x^{\alpha-1} - \alpha x^{\alpha-1}\ln x}{x^{2\alpha}} =$$

$$= \frac{1 - \alpha \ln x}{x^{\alpha+1}}$$

x	0_+		$e^{1/\alpha}$	
f_α'		$+$	0	$-$

$$f_\alpha''(x) = \frac{-\alpha \cancel{x^\alpha} - (\alpha+1)\cancel{x^\alpha} + \alpha(\alpha+1)\cancel{x^\alpha}\ln x}{x^{2\alpha+2}} =$$

$$= \frac{-(2\alpha+1) + \alpha(\alpha+1)\ln x}{x^{\alpha+2}}$$

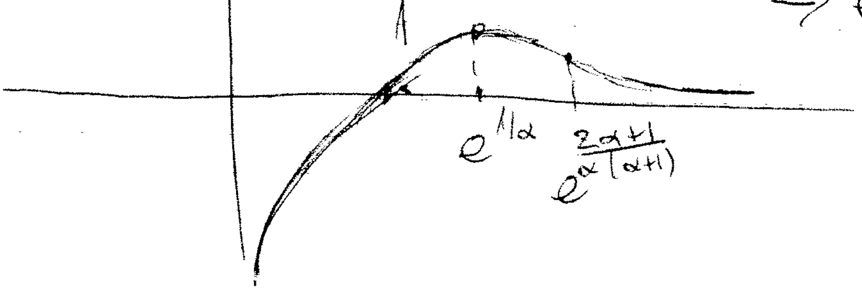
x	0_+		$e^{\frac{2\alpha+1}{\alpha(\alpha+1)}}$	
f_α''		$-$	0	$+$

$\frac{2\alpha+1}{\alpha(\alpha+1)} > 0$

$\Rightarrow$ (alla) f_α har lok. max i $e^{1/\alpha}$, inflexionspkt i $e^{\frac{2\alpha+1}{\alpha(\alpha+1)}} > 1$ (efter nollstället). $f_{\alpha,max} = f_\alpha(e^{1/\alpha}) = 1/\alpha e$

När α växer går f_α snabbare mot 0 i ∞; $f_{\alpha,max}$ minskar, $x_{max} = e^{1/\alpha}$ flytter sig närmare 1 (alltid > 1), pkten $\frac{2\alpha+1}{\alpha(\alpha+1)}$ minsker också, s.a. $e^{\frac{2\alpha+1}{\alpha(\alpha+1)}}$ flytter sig närmare 1; $\frac{2\alpha+1}{\alpha(\alpha+1)} > \frac{1}{\alpha}$ alltid.

$\Rightarrow e^{\frac{2\alpha+1}{\alpha(\alpha+1)}} > e^{1/\alpha}$



4. (a) $\int \ln(x^2+a^2)\,dx =$

$$\overset{p.i.}{=} x\ln(x^2+a^2) - \int x\cdot\frac{1}{x^2+a^2}\cdot 2x\,dx =$$

$$= x\ln(x^2+a^2) - \int \frac{2x^2+2a^2}{x^2+a^2}\,dx + 2a^2\int\frac{dx}{x^2+a^2} =$$

$$= x\ln(x^2+a^2) - 2x + 2\cancel{a^2}\cdot\frac{1}{\cancel{a^2}}\cdot a\int\frac{\frac{1}{a}\,dx}{1+\left(\frac{x}{a}\right)^2} =$$

$$= x\ln(x^2+a^2) - 2x + 2a\cdot\arctan\frac{x}{a}$$

(b) $\int_1^3 \frac{1+\sqrt[3]{2x}}{\sqrt{2x}}\,dx = \left[\begin{matrix}2x=t^6\\ dx=3t^5\,dt\end{matrix}\right] =$

$$= \int_{\sqrt[6]{2}}^{\sqrt[6]{6}} \frac{1+t^2}{\cancel{t^3}}\cdot 3t^{\cancel{5}\,2}\,dt = \left[\cancel{3}\frac{t^3}{\cancel{3}} + \frac{3t^5}{5}\right]_{\sqrt[6]{2}}^{\sqrt[6]{6}} =$$

$$= \sqrt{6} - \sqrt{2} + \frac{3}{5}\sqrt[6]{6^5} - \frac{3}{5}\sqrt[6]{2^5}$$

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5. Konvergent

$$\left.\begin{matrix} a_n < n\cdot\frac{1}{n+1} < 1 \\ a_n > n\cdot\frac{1}{2n} = \frac{1}{2}\end{matrix}\right\} \Rightarrow \text{följden begränsad}$$

$$a_{n+1} - a_n = \cancel{\frac{1}{n+2}} + \dots + \cancel{\frac{1}{2n}} + \frac{1}{2n+1} + \frac{1}{2n+2} -$$

$$-\frac{1}{n+1} - \cancel{\frac{1}{n+2}} - \dots - \cancel{\frac{1}{2n}} =$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \Rightarrow \text{följden växande}$$

$\Rightarrow$ konvergent

(6) Exemplet: $\arcsin x$

begränsad i $(-1, 1)$

(har värden i $(-\frac{\pi}{2}, \frac{\pi}{2})$),

men derivatan växer mot ∞ när

$x \to \pm 1$

Beviset: Antag att f' begränsad i (a, b), $|f'| \leq C$ i (a, b)

Ur medelvärdessatsen följer att

$f(x) = f(x_0) + f'(\xi)(x - x_0)$, där vi väljer $a < x_0 < x < b$; $x_0 < \xi < x$ enligt m.v.s.

$\Rightarrow |f(x)| \leq |f(x_0)| + |f'(\xi)|(x - x_0) \leq$

$\leq |f(x_0)| + C(b - x_0)$

$\Rightarrow f$ begränsad i $[x_0, b)$

analogt, f begränsad i $(a, x_0]$

$\Rightarrow f$ begränsad i (a, b)

Motsägelse!

$\Rightarrow f'$ obegränsad i (a, b)

(8b) Derivatan är

$e^{(\ln x)^2} \cdot \frac{1}{x}$.