TMA970 Inledande matematisk analys F1 & TM1

Lösningar 24/8-15

1. (a) konvergent; (b) divergent; (c) konvergent; (d) falskt; (e) falskt; (f) sant.

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2. (a) $\dfrac{2^{n+1}+3^{n+1}}{2^n+3^n} = \dfrac{3^{n+1}\left(\left(\frac{2}{3}\right)^{n+1}+1\right)}{3^n\left(\left(\frac{2}{3}\right)^n+1\right)} \xrightarrow[n\to\infty]{} \dfrac{3\cdot 1}{1} = 3$

(b) $\dfrac{\ln(1+e^x)}{x} = \dfrac{\ln\left(e^x(1+e^{-x})\right)}{x} =$

$= \dfrac{x+\ln(1+e^{-x})}{x} = 1 + \underbrace{\dfrac{\ln(1+e^{-x})}{e^{-x}}}_{\to 1,\ x\to\infty}\cdot\underbrace{\dfrac{1}{e^x\cdot x}}_{\to 0,\ x\to\infty}$

$\xrightarrow[x\to\infty]{} 1$

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3. D_f : $x^2-1>0 \iff |x|>1$

$\Rightarrow D_f = (-\infty,-1)\cup(1,\infty)$

$f(x)\geq 0 \ \forall x\in D_f$; $0\notin D_f$

$\Rightarrow f(x)\neq 0$ i D_f $\Rightarrow f(x)>0 \ \forall x\in D_f$

f jämn, eftersom $f(-x)=f(x)$

(ej periodisk)

räcker att rita grafen i $(1,\infty)$ och spegla den i y-axeln

$\lim_{x\to 1_+} f(x) = "\dfrac{1}{+0}" = +\infty$; $\lim_{x\to\infty} f(x) = +\infty$

$\Rightarrow$ $x=1$ (och $x=-1$) vertikal asymptot

Sneda?

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{\cancel{x}^{\,1}}{\underbrace{\cancel{|x|}}_{x>0}\sqrt{1-\frac{1}{x^2}}}$$

$$= 1$$

$$\lim_{x\to\infty} (f(x) - 1\cdot x) = \lim_{x\to\infty} \frac{x^2 - x\sqrt{x^2-1}}{\sqrt{x^2-1}} =$$

$$= \lim_{x\to\infty} \frac{(x^2 - x\sqrt{x^2-1})(x^2+x\sqrt{x^2-1})}{\sqrt{x^2-1}\cdot(x^2+x\sqrt{x^2-1})} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x^4} - \cancel{x^4} + x^2}{\sqrt{x^2-1}\cdot x^2\left(1+\sqrt{1-\frac{1}{x^2}}\right)} = 0$$

$\Rightarrow$ den räta linjen med ekvation $y=x$ är sned asymptot i $+\infty$ ($y=-x$ i $-\infty$),

$$f'(x) = \frac{2x\sqrt{x^2-1} - x^2\cdot\frac{1}{\cancel{2}\sqrt{x^2-1}}\cdot\cancel{2}x}{x^2-1} =$$

$$= \frac{2x(x^2-1) - x^3}{(x^2-1)^{3/2}} = \frac{x^3-2x}{(x^2-1)^{3/2}}$$

$x^3-2x = x(x-\sqrt{2})(x+\sqrt{2})$

$x=0 \notin D_f \Rightarrow f'=0$ för $x=\pm\sqrt{2}$

$f'(x) > 0$ $\forall x < -\sqrt{2}$ och $\forall x > \sqrt{2}$

$\quad$ —"— < 0 $\forall x \in (-\sqrt{2},-1) \cup (1,\sqrt{2})$

$\Rightarrow$ f har lok. min i $\pm\sqrt{2}$

$$f''(x) = \frac{(3x^2-2)(x^2-1)^{3/2} - (x^3-2x)\cdot\frac{3}{\cancel{2}}\sqrt{x^2-1}\cdot\cancel{2}x}{(x^2-1)^3} =$$

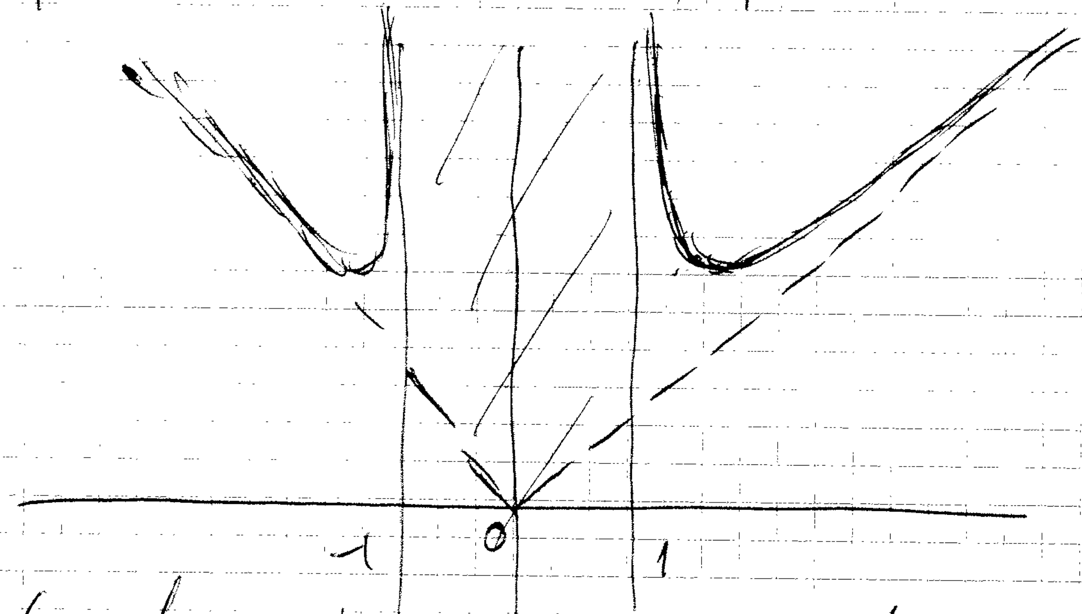
$$= \frac{1}{(x^2-1)^3}\cdot\sqrt{x^2-1}\,\left((3x^2-2)(x^2-1) - 3x(x^3-2x)\right) =$$

$$= \frac{1}{\left(\sqrt{x^2-1}\right)^5}\left(\cancel{3x^4} - 3x^2 - 2x^2 + 2 - \cancel{3x^4} + 6x^2\right) =$$

$$= \frac{x^2+2}{\left(\sqrt{x^2-1}\right)^5} > 0 \quad \forall x \in D_f$$

$\Rightarrow$ f konvex i $(-\infty,-1)$ och i $(1,\infty)$

x	$-\infty$		$-\sqrt{2}$		-1		1		$\sqrt{2}$		$+\infty$
f	$y=-x$	$\searrow$	2	$\nearrow$	$+\infty$		$+\infty$	$\searrow$	2	$\nearrow$	$y=x$
f'		$-$ $-$	0	$+$				$-$	0	$+$	$+$
f''		$+$ $+$		$+$				$+$		$+$	$+$



(grafen ska vara symmetrisk m.a.p. y-axeln)

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④ (a) $\displaystyle\int \frac{dx}{\sqrt{7-5x^2}} = \frac{1}{\sqrt{7}}\int \frac{dx}{\sqrt{1-\frac{5}{7}x^2}} = \frac{1}{\sqrt{5}}\int \frac{\sqrt{\frac{5}{7}}\,dx}{\sqrt{1-\left(\sqrt{\frac{5}{7}}x\right)^2}} =$

$$= \frac{1}{\sqrt{5}} \arcsin\left(\sqrt{\frac{5}{7}}\,x\right) \quad (+C)$$

$$(b)\ I = \int_0^1 e^{-x}\sin 2x = \qquad \boxed{4}$$

$$= -\int_0^1 (e^{-x})' \sin 2x\,dx \overset{p.i.}{=} -\left[e^{-x}\sin 2x\right]_0^1 +$$

$$+ \int_0^1 e^{-x}\cdot 2\cos 2x\,dx = -\frac{\sin 2}{e} -$$

$$-\left[e^{-x}\cdot 2\cos 2x\right]_0^1 - \underbrace{\int_0^1 e^{-x}\cdot 4\sin 2x\,dx}_{=4I}$$

$$\Rightarrow \int_0^1 e^{-x}\sin 2x\,dx = \frac{1}{5}\left(2 - \frac{\sin 2 + 2\cos 2}{e}\right)$$

(5.)

$$\frac{f(x)-f(1)}{x-1} = \frac{\sqrt[3]{x-1} - 0}{x-1} =$$

$$= \frac{1}{\sqrt[3]{(x-1)^2}} \xrightarrow[x\to 1]{} \infty$$

$\Rightarrow$ f har ej ändlig derivata i 1

(6.) $\tan x$ är definierad i alla intervall av typen

$$\left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right), \forall k \in \mathbb{Z}$$

(och odefinierad i $(2k+1)\frac{\pi}{2}$ $\forall k \in \mathbb{Z}$)

$$\lim_{x\to (2k-1)\frac{\pi}{2}+} \tan x = -\infty, \quad \lim_{x\to(2k+1)\frac{\pi}{2}-} \tan x = +\infty \ \forall k \in \mathbb{Z}$$

$$\Rightarrow \lim_{(2k-1)\frac{\pi}{2}+} (\tan x - x) = -\infty; \quad \lim_{(2k+1)\frac{\pi}{2}-} (\tan x - x) = +\infty$$

$\Rightarrow$ $\exists\ \varepsilon_1, \varepsilon_2 > 0$ s.a. $(2k-1)\frac{\pi}{2} + \varepsilon_1 < (2k+1)\frac{\pi}{2} - \varepsilon_2$

s.a. $\tan x - x < 0$ i $\left((2k-1)\frac{\pi}{2} + \varepsilon_1,\right)$

$\tan x - x > 0$ i $(2k+1)\frac{\pi}{2} - \varepsilon_2$

$\Rightarrow$ $\tan x - x$ har nollställe i intervallet $\left((2k-1)\frac{\pi}{2} + \varepsilon_1, (2k+1)\frac{\pi}{2} - \varepsilon_2\right)$

$\forall\ k \in \mathbb{Z}$

två sådana intervall har tomt snitt

$\Rightarrow$ $\tan x - x$ har oändligt många reella nollställen

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8b

$$\int \frac{x}{(x^2+2x+2)^2}\,dx = \int \frac{x}{((x+1)^2+1)^2}\,dx =$$

$$= \underbrace{\int \frac{x+1}{((x+1)^2+1)^2}\,dx}_{=\frac{1}{2}\ln(x^2+2x+2)} - \underbrace{\int \frac{dx}{((x+1)^2+1)^2}}_{\text{enligt boken}}$$