TMA 970 Inledande matematisk analys F1 / TM1

Lösningar 7/1-2016

1. (a) divergent; (b) konvergent; (c) konvergent; (d) falskt; (e) sant; (f) sant.

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2. (a) $(x>0)$

$$\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}} = \sqrt{1+\frac{1}{x}\sqrt{x+\sqrt{x}}} =$$

$$= \sqrt{1+\underbrace{\sqrt{\frac{1}{x}+\frac{1}{x^2}\sqrt{x}}}_{\to 0}} \xrightarrow[x\to\infty]{} 1$$

(b) $\ln\cos ax = \ln(1+(\cos ax - 1)) =$

$$= \frac{\ln(1+(\cos ax-1))}{\cos ax - 1}(\cos ax - 1)$$

$$\frac{\ln\cos 3x}{\ln\cos 2x} = \frac{\ln(1+(\cos 3x-1))}{\cos 3x - 1}\cdot\frac{\cos 2x - 1}{\ln(1+(\cos 2x-1))}\cdot$$

$$\cdot\frac{\cos 3x - 1}{\cos 2x - 1} =$$

$$= \underbrace{\frac{\ln(1+(\cos 3x-1))}{\cos 3x-1}}_{\xrightarrow[x\to 0]{} 1}\cdot\underbrace{\frac{\cos 2x-1}{\ln(1+(\cos 2x-1))}}_{\xrightarrow[x\to 0]{} 1}\cdot\frac{1-\cos 3x}{1-\cos 2x} \to ?$$

ty $\cos 3x - 1 \to 0$, $\cos 2x - 1 \to 0$

$\triangle 2$

$$\frac{1-\cos 3x}{1-\cos 2x} = \frac{1-\cos^2 3x}{1-\cos^2 2x} \cdot \frac{1+\cos 2x}{1+\cos 3x} =$$

$$= \frac{\sin^2 3x}{(3x)^2} \cdot \frac{(2x)^2}{\sin^2 2x} \cdot \frac{9\cancel{x^2}}{4\cancel{x^2}} \cdot \frac{1+\cos 2x}{1+\cos 3x}$$

$$\xrightarrow[x\to 0]{} 1 \cdot 1 \cdot \frac{9}{4} \cdot \frac{2}{2}$$

$$\Rightarrow \frac{\ln \cos 3x}{\ln \cos 2x} \xrightarrow[x\to 0]{} \frac{9}{4}$$

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(3) $f(x) = \frac{x^2-1}{x^3}$ $\quad D_f: x \neq 0$

Nollställen: ± 1

$f > 0$ för $\begin{cases} |x| > 1 \\ x > 0 \end{cases}$; $f < 0$ för $\begin{cases} |x| < 1 \\ x > 0 \end{cases}$

$f(-x) = -f(x) \Rightarrow f$ udda

ej periodisk

Räcker att ~~titta~~ på $x > 0$.

$$\lim_{x\to 0+} f(x) = \text{"}\frac{-1}{+0}\text{"} = -\infty$$

$$\lim_{x\to\infty} f(x) = 0$$

$\Rightarrow$ $x=0$ vertikal asymptot

$y=0$ horisontell asymptot i $\pm\infty$

$$f'(x) = \frac{2x \cdot x^{\cancel{3}} - 3\cancel{x^2}(x^2-1)}{x^{\cancel{6}4}} = \frac{3-x^2}{x^4}$$

x		$-\sqrt{3}$		0		$\sqrt{3}$	
f'	$-$	0	$+$	|	$+$	0	$-$

$\Rightarrow$ f har lok. max i $\sqrt{3}$
(och lok. min i $-\sqrt{3}$)

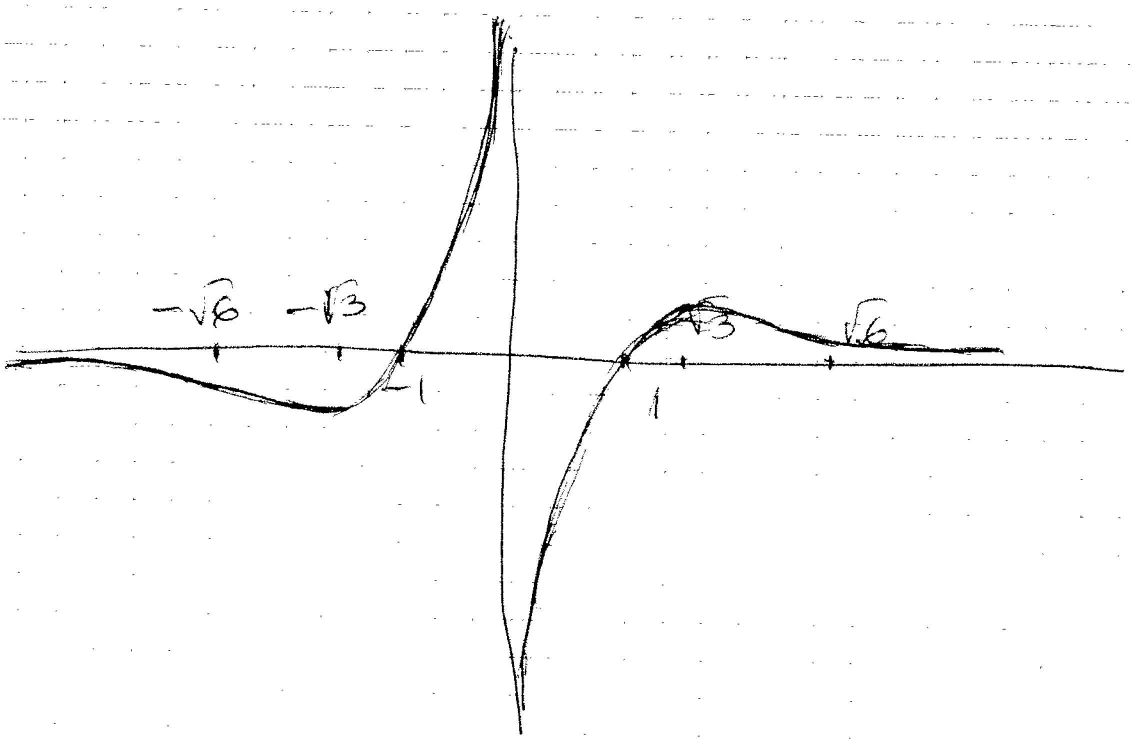
$$f''(x) = \frac{-2x\cdot x^{\cancel{4}} - 4\cancel{x^{3}}(3-x^{2})}{x^{\cancel{8}\,5}} = \frac{2x^2-12}{x^5}$$

x	$-\sqrt{6}$		0		$\sqrt{6}$	
f''	$-$ 0 $+$		|		$-$ 0 $+$	

$\Rightarrow$ inflexion för $x=\sqrt{6}$ (och $x=-\sqrt{6}$)

f konvex i $(-\sqrt{6},0)$ & i $(\sqrt{6},\infty)$
konkav i $(-\infty,-\sqrt{6})$ och i $(0,\sqrt{6})$

x	$-\infty$	$-\sqrt{6}$	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	$\sqrt{6}$	$+\infty$
f	$\to 0$	infl.	lok. min	0	$+\infty$ | $-\infty$	0	lok. max	infl.	$\to 0$
f'		$-$	0	$+$	|	$+$	0	$-$	
f''	$-$	0		$+$	|	$-$		0	$+$



④ (a) f rationell funktion

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+2}$$

$$x = A(x-1)(x^2+2x+2) + B(x^2+2x+2) + $$
$$+ (Cx+D)(x-1)^2$$

$x=1$: $\quad 1 = 5B \quad \Rightarrow \quad B = 1/5$

x^3 : $\quad 0 = A + C \quad \Rightarrow \quad C = -A$

x^0 : $\quad 0 = -2A + 2B + D$

x^2 : $\quad 0 = A + B - 2C + D$

$$-2A + \frac{2}{5} + D = 0$$
$$3A + \frac{1}{5} + D = 0$$

$$5A - \frac{1}{5} = 0 \Rightarrow A = \frac{1}{25}, \quad C = -\frac{1}{25}$$
$$D = -\frac{8}{25}$$

$$\int \frac{dx}{x-1} = \ln|x-1| \quad (+C)$$

$$\int \frac{dx}{(x-1)^2} = -\frac{1}{x-1} \quad (+C)$$

$$\int \frac{Cx+D}{x^2+2x+2}\,dx = \int \frac{C(x+1)+D-C}{(x+1)^2+1}\,dx$$

$$\int \frac{x+1}{(x+1)^2+1}\,dx = \frac{1}{2}\int \frac{((x+1)^2+1)'}{(x+1)^2+1}\,dx =$$

$$= \frac{1}{2}\ln(x^2+2x+2) \quad (+C)$$

$$\int \frac{dx}{(x+1)^2+1} = \arctan(x+1) \quad (+C)$$

återstår att plocka ihop

(b) $\int_0^1 1 \cdot \arccos x \, dx =$ $\boxed{3}$

$= \Big[x \arccos x\Big]_0^1 - \int_0^1 \frac{x}{-\sqrt{1-x^2}} \, dx =$

$= 1 \cdot 0 - 0 \cdot \frac{\pi}{2} + \frac{1}{2} \int_0^1 \frac{(x^2)'}{\sqrt{1-x^2}} \, dx =$

$= 0 - \frac{1}{2} \int_0^1 \frac{(1-x^2)'}{\sqrt{1-x^2}} \, dx = \left[-\frac{1}{2} \cdot \frac{(1-x^2)^{1/2}}{1/2} \right]_0^1 =$

$= -0 + 1 = \underline{\underline{1}}$

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5. Vi behöver titta på konvergens/divergens i 0, 1 och ∞.

$x_0 = 0$: $\int_0^{1/2} \frac{dx}{x^\alpha |x-1|^\beta} = \lim_{\varepsilon \to 0} \int_\varepsilon^{1/2} \frac{dx}{x^\alpha |x-1|^\beta}$

$|x-1|^\beta \xrightarrow[x \to 0]{} 1$

$\Rightarrow \quad \frac{1}{2} < |x-1|^\beta < \frac{3}{2}$ för tillräckligt små $\underline{x}$

$\Rightarrow \frac{1}{2x^\alpha} < \frac{1}{x^\alpha |x-1|^\beta} < \frac{3}{2x^\alpha}$ för små $\underline{x}$

$\Rightarrow$ (i) för $\alpha < 1$ $\quad 0 \leq \frac{1}{x^\alpha |x-1|^\beta} < \frac{3}{2} \cdot \frac{1}{x^\alpha}$

$\Rightarrow$ konvergens

(ii) för $\alpha \geq 1$ $\quad \frac{1}{x^\alpha |x-1|^\beta} > \frac{1}{2} \cdot \frac{1}{x^\alpha} \; (>0)$

$\Rightarrow$ divergens

$\Rightarrow$ integralen konvergent i 0 omm $\alpha < 1$.

$x_1 = 1$: exakt samma förfarande, (6)
fast man jämför med
$\frac{1}{(1-x)^\beta}$ i $\left(\frac{1}{2}, 1\right)$
och med $\frac{1}{(x-1)^\beta}$ i $(1, 2)$

konvergens omm $\beta < 1$

∞ : $\frac{1}{x^\alpha |x-1|^\beta} = \frac{1}{x^\alpha (x-1)^\beta} =$

$= \frac{1}{x^{\alpha+\beta}\left(1-\frac{1}{x}\right)^\beta}$ $\quad \left(1-\frac{1}{x}\right)^\beta \xrightarrow[x\to\infty]{} 1$

$\Rightarrow \frac{1}{2} \cdot \frac{1}{x^{\alpha+\beta}} < \frac{1}{x^{\alpha+\beta}\left(1-\frac{1}{x}\right)^\beta} < \frac{3}{2} \cdot \frac{1}{x^{\alpha+\beta}}$

för tillräckligt stora x

$\Rightarrow$ resonemang som ovan ger
att konvergens i ∞ uppträder
omm $\alpha+\beta > 1$

$\Rightarrow$ integralen konvergent omm $\begin{cases} \alpha < 1 \\ \beta < 1 \\ \alpha+\beta > 1 \end{cases}$

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(6) $\lim_{x\to\infty} f(x) = l \in \mathbb{R}$

$\Rightarrow \exists A : \forall x > A \quad l-1 < f(x) < l+1$

f kontinuerlig i $[a, A] \Rightarrow \exists C \in \mathbb{R}$:
(Weierstraß) $|f(x)| \leq C \quad \forall x \in [a, A]$

$$\Rightarrow |f(x)| \leq C + |l-1| + |l+1| \quad \forall x \in [a, \infty)$$

(8.) $\left(\sin\sqrt{x^2+1}\right)' = \cos\sqrt{x^2+1} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$, enligt kedjeregeln

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