TMA970 Inledande matematisk analys F/TM

Lösningar 22/12-16

1. (a) divergent; (b) divergent; (c) divergent; (d) sant; (e) falskt; (f) sant.

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2. (a)
$$\frac{e^{\sin^2 x}-\cos x}{\sin^2 x} = \frac{e^{\sin^2 x}-1}{\sin^2 x} + \frac{1-\cos x}{\sin^2 x} =$$
$$= \frac{e^{\sin^2 x}-1}{\sin^2 x} + \frac{\cancel{1-\cos^2 x}}{\cancel{\sin^2 x}\cdot(1+\cos x)} \xrightarrow[x\to 0]{} 1+\frac{1}{2},$$
då $\sin^2 x \xrightarrow[x\to 0]{} 0$.

(b)
$$\sqrt{x^3-x^2+1}-\sqrt{x^3+x^2+1} =$$
$$= \frac{\cancel{x^3}-x^2+\cancel{1}-\cancel{x^3}-x^2-\cancel{1}}{\sqrt{x^3-x^2+1}+\sqrt{x^3+x^2+1}} = \frac{-2x^2}{|x|^{3/2}\left(\sqrt{1-\frac{x^2-1}{x^3}}+\sqrt{1+\frac{x^2+1}{x^3}}\right)}$$
$$\overset{x\to\infty}{=} \frac{-2x^{1/2}}{\left(\sqrt{1-\frac{x^2-1}{x^3}}+\sqrt{1+\frac{x^2+1}{x^3}}\right)} \longrightarrow \text{"}\frac{-2\cdot\infty}{2}\text{"} = -\infty$$

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3. $f(x)=\sqrt{x^3+1}$ $\quad D_f:\ x^3\geq -1 \iff x\geq -1$

Nollställe: $x^3+1=0 \iff x=-1$

$f(x)\geq 0\quad \forall x\in D_f \Rightarrow$ min i 0

$D_f = [-1, \infty)$

$\lim\limits_{x\to\infty} f(x) = \infty$

Inga vertikala asymptoter

Sned?

$$\lim_{x\to\infty} \frac{f(x)}{x} = \frac{\sqrt{x^3+1}}{x} = \frac{x^{\cancel{3/2}\,1/2}\sqrt{1+\frac{1}{x^3}}}{\cancel{x}} \xrightarrow[x\to\infty]{} \infty$$

$\Rightarrow$ ingen sned asymptot i ∞

Inga symmetrier

$$f'(x) = \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2, \text{ def. i } (-1,\infty)$$

$f' = 0$ endast i $x_0 = 0 \in (-1,\infty)$

x	-1		0	
f'		$+$	0	$+$

tecknet samma före och efter 0

$\Rightarrow$ inget lokalt extremum i 0

~~terasspkt~~

f växande i hela D_f

$\lim\limits_{x\to -1} f' = \infty$

$\Rightarrow$ vertikal tangent till f:s graf i $(-1, 0)$

$$f''(x) = \frac{3}{2} \cdot \frac{2x\sqrt{x^3+1} - x^2\frac{3x^2}{2\sqrt{x^3+1}}}{x^3+1} =$$

$$= \frac{3}{2} \cdot \frac{4x(x^3+1) - 3x^4}{2\left(\sqrt{x^3+1}\right)^3} = \frac{3}{2}\,\frac{x(x^3+4)}{2(x^3+1)^{3/2}}$$

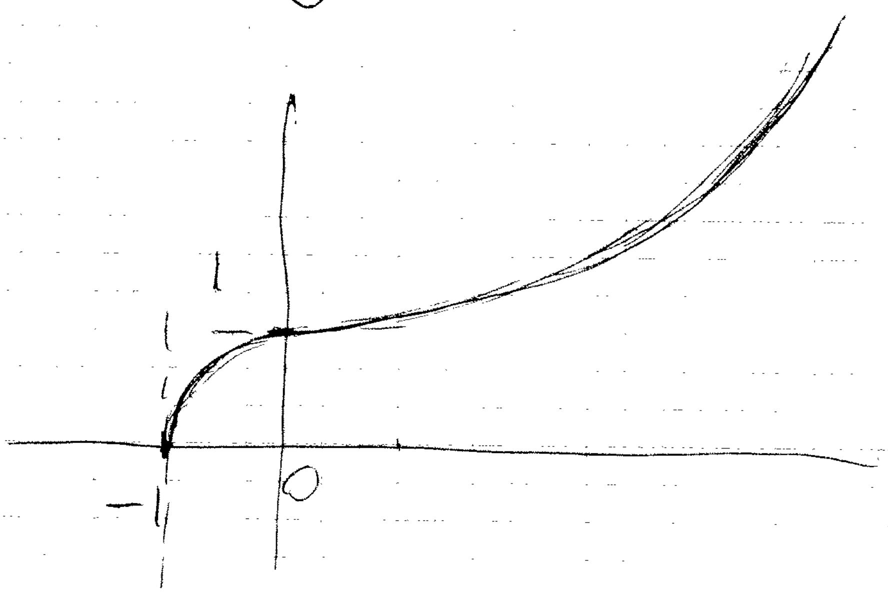
$x^3+4>0$ i D_f

$\Rightarrow f''=0$ endast i $x_0=0$

x	-1		0	
f''		$-$	0	$+$

inflexion för $x=0$ (vilket vi visste)

x	-1		0		$+\infty$
f	0 min	$\nearrow$	1	$\nearrow$	$+\infty$
f'	$+\infty$	$+$	0	$+$	
f''		$-$	0	$+$	



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(4) (a) $x>0$

$$\int \frac{dx}{x\sqrt{2-x^2}} = \int \frac{dx}{x^2\sqrt{\frac{2}{x^2}-1}} =$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{x^2\sqrt{\frac{1}{x^2}-\frac{1}{2}}} = \left[\begin{array}{l} t=\frac{1}{x} \\ dt=-\frac{1}{x^2}dx \end{array}\right] =$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2-\frac{1}{2}}} = -\frac{1}{\sqrt{2}} \ln\left|t+\sqrt{t^2-\tfrac{1}{2}}\right| + C$$

eller (ty $t>0$?

$= -\frac{1}{\sqrt{2}} \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - \frac{1}{2}} \right) + C$ (4)

$x < 0$: $\sqrt{x^2} = -x$, f.ö. samma

Alternativt (standardlösning):

$$x = \sqrt{2} \sin s \qquad dx = \sqrt{2} \cos s \, ds$$

$$\int \frac{dx}{x\sqrt{2-x^2}} = \frac{\sqrt{2}}{\sqrt{8}} \int \frac{\cos s}{\sin s \, \sqrt{2}\, |\cos s|} \, ds$$

$\cos s > 0$: $\frac{1}{\sqrt{2}} \int \frac{ds}{\sin s}$

löses t.ex. med standard-substitutionen

$\cos s < 0$: motsvarande, med $-$

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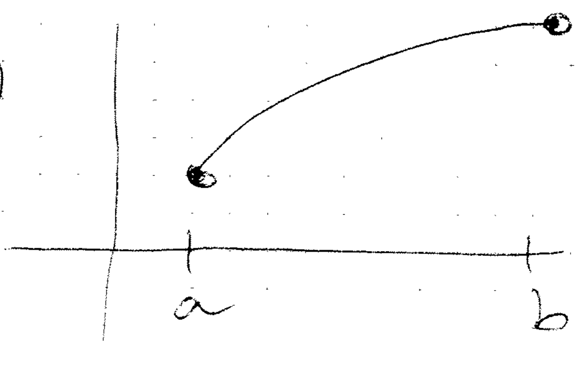
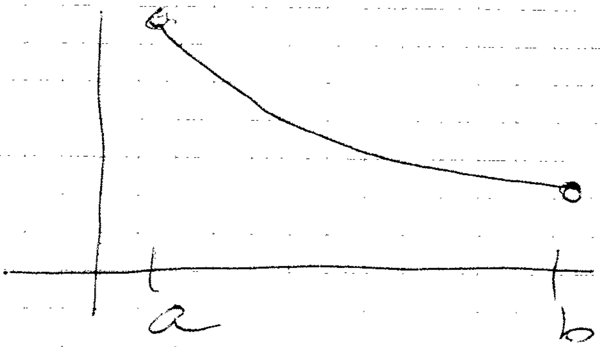
(4)(b) $$\int_0^1 x e^{-3x} \, dx = -\frac{1}{3} \int_0^1 x \left(e^{-3x}\right)' dx \overset{p.i.}{=} \left[-\frac{1}{3} x e^{-3x} \right]_0^1 + \frac{1}{3} \int_0^1 1 \cdot e^{-3x} \, dx =$$

$$= -\frac{1}{3} e^{-3} - \frac{1}{9} \left[e^{-3x} \right]_0^1 = -\frac{4}{9} e^{-3} + \frac{1}{9}$$

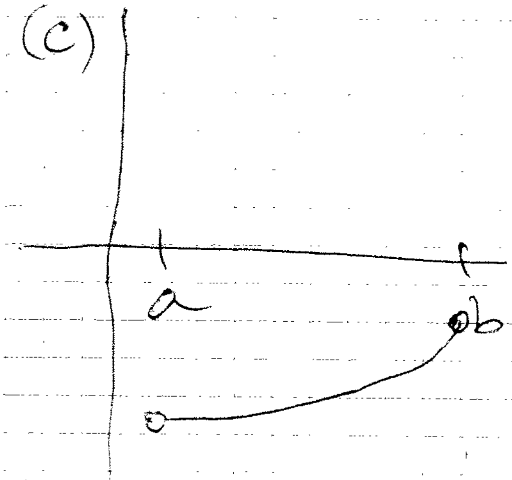
(5.) (a,b) : (a)



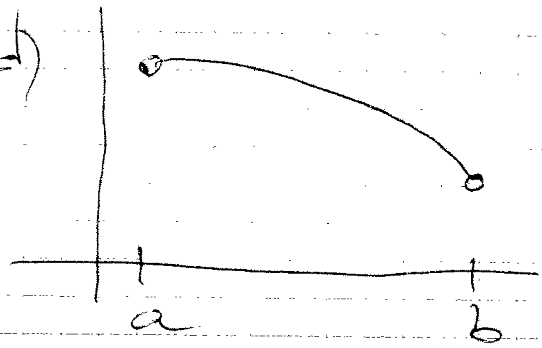
(b)



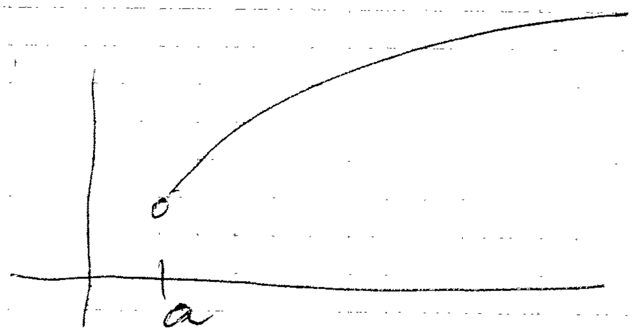
(c)



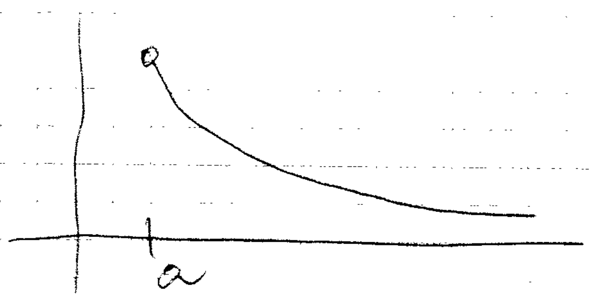
(d)



(a, ∞) : (a)



(b)



(c) & (d) omöjliga

(6.) $n = 1$: ett reellt nollställe för $p \neq -1$; inget om $p = -1$

$n \geq 2$: f' måste ha ett nollställe mellan två nollställen för f

$$f'(x) = (n)\,x^{n-1} + p = n x^{n-1} + p$$

har ett nollställe för n jämnt och max två för n udda

$\Rightarrow$ f har max två för n jämnt, max tre för n udda