Holger Broman, August 19, 1999

Application Example: Controllability and Observability

In control engineering you often represent a system under study in a state-space form.

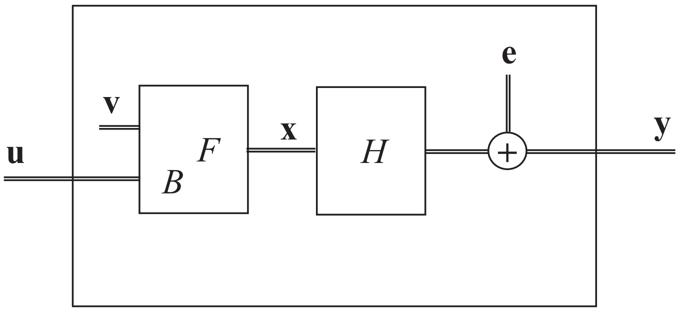

Figure 1. A system in state-space form.

The following two equations describe the system:

$$\begin{cases} \mathbf{x}(t) = F\mathbf{x}(t-1) + B\mathbf{u}(t-1) + \mathbf{v}(t-1) \\ \mathbf{y}(t) = H\mathbf{x}(t) + \mathbf{e}(t), \end{cases} \tag{1}$$

and the notation is

$\mathbf{u}$	a K-dimensional input signal
$\mathbf{x}$	the N-dimensional state vector of the system
$\mathbf{v}$	an N-dimensional disturbance vector, "system noise"
F	the system matrix, $N \times N$
B	an $N \times K$ matrix
$\mathbf{y}$	an L-dimensional output signal
$\mathbf{e}$	an L-dimensional disturbance, "measurement noise"
H	the observation matrix, $N \times L$

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Now, controllability concerns the following question[1]:

Given a system at rest, $\mathbf{x}(0) = 0$, and in the absence of noise, $\mathbf{v}(t) = 0\ \forall\ t$, can you force $\mathbf{x}(t)$ to an arbitrary position in $\mathbb{R}^N$ by a proper choice of the input sequence $\mathbf{u}(0), \ldots, \mathbf{u}(t-1)$?

[1] Actually, the problem formulated should be called reachability or controllability from the origin. Most often it is simply called controllability.

Let us see what happens:

$$\begin{aligned}
\mathbf{x}(1) =& B\,\mathbf{u}(0) \\
\mathbf{x}(2) =& F\,\mathbf{x}(1) + B\,\mathbf{u}(1) = F\,B\,\mathbf{u}(0) + B\,\mathbf{u}(1) \\
\vdots& \\
\mathbf{x}(t) =& F^{t-1}B\,\mathbf{u}(0) + \cdots + B\,\mathbf{u}(t-1) = \\
=& \left[F^{t-1}B \; F^{t-2}B \; \cdots \; B\right] \left[\mathbf{u}^T(0) \; \mathbf{u}^T(1) \; \cdots \; \mathbf{u}^T(t-1)\right]^T
\end{aligned}$$

As you are free to choose the input sequence freely, the question of controllability boils down to the question concerning the range-space of the matrix

$$C_t = \left[F^{t-1}B \; F^{t-2}B \; \cdots \; B\right].$$

We make the following observations:

$$\mathrm{Rank}(C_t) \leq \mathrm{Rank}(C_{t+1}) \leq N$$

as C_t has N rows, and going from t to $t+1$ gives you more columns so that the rank is non-decreasing. There is no need to study instances $t > N$, due to the Cayley-Hamilton theorem, which states that any matrix satisfies its own characteristic function. Here is the proof:

Let λ be an eigenvalue of F. Then

$$\det(F - \lambda\, I) = 0$$

$$\Longleftrightarrow$$

$$f_N \lambda^N + \ \ldots \ + f_0 = 0 \text{ for some set } f_n, \ n = 0, \ldots, N$$

Multiply by the corresponding eigenvector from the right, and replace $\lambda\,\mathbf{g}$ by $F\,\mathbf{g}$. Proceed until no λ's remain.

$$\left(f_N\, F^N + \ \ldots \ + f_0\, I\right)\mathbf{g} = 0,$$

and we conclude

$$f_N\, F^N + \ \ldots \ + f_0\, I = 0,$$

which proves the theorem.

Now, returning to the original problem, we see that going beyond $t = N$ will not increase the rank of C_t, as you add columns that are linearly dependent on already existing ones. We have now proven the following:

> The system described by equation (1) is controllable if and only if the controllability matrix $C = \left[F^{N-1}B \;\; F^{N-2}B \; \cdots \; FB \;\; B\right]$ has full rank.

Note 1: This is by no means trivial, as in most applications $\dim \mathbf{u} < \dim \mathbf{x}$.

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Observability concerns the following. Given a system without noise, $\mathbf{v}(t) \equiv 0$ and $\mathbf{e}(t) \equiv 0$, and with no input, $\mathbf{u} \equiv 0$, is there any initial position $\mathbf{x}(0) = \mathbf{s}$ such that the system is 'silent', i.e. $\mathbf{y}(t) = 0$, $t = 0, 1, \ldots$. Here we go:

$$\begin{aligned}
\mathbf{y}(0) =& H\,\mathbf{x}(0) \\
\mathbf{y}(1) =& H\,\mathbf{x}(1) = HF\,\mathbf{x}(0) \\
&\vdots \\
\mathbf{y}(t) =& H\,\mathbf{x}(t) = HF^t\,\mathbf{x}(0)
\end{aligned}$$

Put in vector form:

$$\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(t) \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^t \end{bmatrix} \mathbf{x}(0) = O_t\,\mathbf{x}(0)$$

Just as in the preceding case, it suffices to study $t = N - 1$ — remember the Cayley-Hamilton theorem. The vector on the left hand side is identically zero iff $\mathbf{x}(0)$ is in the null space of the observability matrix:

> The system described by equation (1) is observable if and only if the observability matrix $O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{N-1} \end{bmatrix}$ has full rank.

Note 2: A system is observable iff there are no silent states.

Note 3: For an observable system, the initial state can be calculated from the observations $\mathbf{y}(0), \mathbf{y}(1), \ldots$.

Application Example: MUSIC

The following example is a version of an algorithm, MUSIC, with reference to problems in communication. The acronym MUSIC stands for MUltiple SIgnal Classification. This is the scenario: we have a scalar signal, $y(t)$, which is the sum of L complex sinusoids in additive noise,

$$y(t) = \sum_{\ell=1}^{L} b_\ell \, e^{j\omega_\ell t} + e(t).$$

The task at hand is to estimate the frequencies ω_ℓ from the observations $y(t)$. We start by stacking some observations in a vector:

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-M) \end{bmatrix} = \text{ use the model of } y(t) =$$

$$= \begin{bmatrix} 1 & & 1 \\ e^{-j\omega_1} & & e^{-j\omega_L} \\ \vdots & \cdots & \vdots \\ \vdots & & \vdots \\ e^{-jM\omega_1} & & e^{-jM\omega_L} \end{bmatrix} \begin{bmatrix} b_1 \, e^{j\omega_1 t} \\ \vdots \\ \vdots \\ \vdots \\ b_L \, e^{j\omega_L t} \end{bmatrix} + \begin{bmatrix} e(t) \\ \vdots \\ \vdots \\ \vdots \\ e(t-M) \end{bmatrix}$$

We note with pleasure that the matrix involved is vandermonde, and thus has full rank for ω_ℓ distinct. Introduce the notations

$$\mathbf{a}(\omega) = \begin{bmatrix} 1 & e^{-j\omega} & \cdots & e^{-jM\omega} \end{bmatrix}^H,$$

and

$$s_\ell = b_\ell \, e^{j\omega_\ell t},$$

so that

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{a}(\omega_1) & \cdots & \mathbf{a}(\omega_L) \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_L \end{bmatrix} + \text{ noise} = A\,\mathbf{s} + \text{ noise}.$$

Now, we make some observations:

- The steering vector $\mathbf{a}(\omega)$ is a curve in $\mathbb{C}^{M+1}$.
- For $M \geq L-1$, $\{\mathbf{a}(\omega_\ell)\}_{\ell=1}^{L}$ span an L-dimensional subspace of $\mathbb{C}^{M+1}$, the signal subspace.
- The set $\{\omega_\ell\}_{\ell=1}^{L}$ is the solution to the intersection of $\mathbf{a}(\omega)$ and the signal subspace. The set is unique as the vandermonde matrix has full rank equal to L.

Note 1: The signal $\sum_{\ell=1}^{L} b_\ell \, e^{j\omega_\ell t}$ is a rank L signal.

In conclusion, it seems to be a good idea to estimate the signal subspace. One way to do this is to estimate the correlation matrix

$$\hat{R} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}(n) \, \mathbf{y}^H(n).$$

As this is a course on linear algebra, we disregard the influence of the noise.

$$\begin{aligned} \hat{R} =& \frac{1}{N} \sum_{n=1}^{N} A \, \mathbf{s}(n) \, \mathbf{s}^H(n) \, A^H = \\ =& A \left[\frac{1}{N} \sum_{1}^{N} \mathbf{s}(n) \, \mathbf{s}^H(n) \right] A^H = A \, \hat{R}_{\mathbf{s}} \, A^H. \end{aligned}$$

Hooray, we may guess that the range-space of $\hat{R}$ equals the range-space of A, equals the signal subspace[2].

Note 2: When noise is present, assume that it is white and small. Then take the L largest eigenvalues of $\hat{R}$ ($\hat{R}$ Hermitian implies real-valued non-negative eigenvalues). The corresponding eigenvectors span the estimate of the signal subspace.

Let S be the matrix of dimensions $(M+1) \times L$ that has orthonormal columns that span the signal subspace. As the norm of $\mathbf{a}(\omega)$ equals $M+1$, independent of ω, the desired solution is found by finding the L maxima to the following:

$$\max_{\omega} \left| \mathbf{a}^H(\omega) \;\, S \right|^2$$

To produce nice plots, you can construct the MUSIC pseudo-spectrum

$$P(\omega) = \frac{1}{1 - \frac{\left|\mathbf{a}^H(\omega) \;\, S\right|^2}{\left|\mathbf{a}(\omega)\right|^2}}$$

Please run the m-file musicapp in Matlab

[2] To really understand this you should take a course in signal processing.

```
% musicapp.m
% a simple application of the MUSIC algorithm
clear

N=200;
sigma=1;
t=(1:N)';
s=sin(5*2*pi/N*t)+sin(23*2*pi/N*t); % a rank 4 signal
data=s+sigma*randn(N,1);

% estimate the autocorrelation matrix
M=19;
R=zeros(M+1,M+1);
for n=1:N-M
R=R+data(n:n+M)*data(n:n+M)';
end
R=R/(N-M);

% calculate the rangespace of R
[U S V]=svd(R);
Neig=4; % the 4 first left eigenvectors
Ssub=U(:,1:Neig); % span the signal subspace.

% calculate the pseudospectrum
m=1;
steer=0:M;
for angle=-pi:pi/100:pi
aflip=exp(-j*angle*steer); % the transposed steering vector for argument angle
dum=(norm(aflip*Ssub))^2/(M+1); % norm(aflip)^2=M+1
P(m)=1/(1-dum); % this screws up if you remove the noise
m=m+1;
end

% plot the data and the pseudospectrum.
figure(1), clf, hold on
plot(data)
title('Two sinusoids in white noise')
figure(2), clf, hold on
plot([-pi:pi/100:pi],P)
title('MUSIC pseudospectrum')

% There are a lot of experiments to be carried out.
% Vary N
% Vary sigma - too much noise and you may miss the signal subspace.
% Vary Neig, we do not always know the rank of the signal subspace
% Vary M, it affects resolution
```

Application Example: ESPRIT

Refer back to the example presenting MUSIC to find the relevant model for the data

$$\mathbf{y}(t) = A\,\mathbf{s}(t) + e(t),$$

where the steering matrix A is

$$A(\omega) = [\mathbf{a}(\omega_1) \;\cdots\; \mathbf{a}(\omega_L)] =$$

$$= \begin{bmatrix} 1 & 1 \\ e^{-j\omega_1} & e^{-j\omega_L} \\ \vdots & \vdots \\ \vdots & \vdots \\ e^{-jM\omega_1} & e^{-jM\omega_L} \end{bmatrix}.$$

Please note the vandermonde structure. Now, partition A in two different ways:

$$A = \begin{bmatrix} A_1 \\ \text{last row} \end{bmatrix} = \begin{bmatrix} \text{first row} \\ A_2 \end{bmatrix}.$$

It follows

$$A_2 = A_1\,\Phi,$$

where

$$\operatorname{diag}(\Phi) = \left(e^{-j\omega_1}, \ldots, e^{-j\omega_L}\right).$$

So, to estimate the frequencies, find Φ.

Unfortunately, we do not have A. We can estimate, however, the signal subspace from the data, and construct the matrix S that spans the same subspace as does A, as demonstrated in the derivation of the MUSIC algorithm. This means that there exists a square full rank transformation matrix that relates A and S:

$$S = A\,T.$$

Now partition S as we did with A. Then

$$S_1 = A_1\,T$$

$$S_2 = A_2\,T,$$

and

$$A_2 = A_1\,\Phi$$

implies

$$S_2\,T^{-1} = S_1\,T^{-1}\,\Phi,$$

which in turn leads to

$$S_2 = S_1\,T^{-1}\,\Phi\,T = S_1\,\Psi.$$

As Ψ and Φ are related by a similarity transform, they have the same eigenvalues, and we find the frequencies by finding the eigenvalues of Ψ. In practice, you need to solve

$$S_2 = S_1 \, \Psi$$

in a least squares sense. In Matlab code, it is really simple:

$$\Psi = S_1 \backslash S_2$$

will do it.

Please run the m-file espritapp in Matlab.

```
% espritapp.m
% a simple application of the ESPRIT algorithm
clear

N=200;
sigma=1;
t=(1:N)';
s=sin(5*2*pi/N*t)+sin(23*2*pi/N*t); % a rank 4 signal
data=s+sigma*randn(N,1);

% estimate the autocorrelation matrix
M=19;
R=zeros(M+1,M+1);
for n=1:N-M
R=R+data(n:n+M)*data(n:n+M)';
end
R=R/(N-M);

% calculate the rangespace of R, and the signal subspace
[U S V]=svd(R);
Neig=4; % the 4 first left eigenvectors
Ssub=U(:,1:Neig); % span the signal subspace.

% now for the ESPRIT version
S1=Ssub(1:M,:);
S2=Ssub(2:M+1,:);
poles=eig(S1\S2);

% plot the poles
figure(1), clf, axis equal, hold on
plot(real(poles),imag(poles),'rp')
t=0:1000;
t=2*pi/1000*t;
plot(cos(t),sin(t),'b',[-1.2 1.2], [0 0],'k',[0 0],[-1.2 1.2],'k')
title('Detected frequencies')
% Check the angles, should be 5 (23) /200 *2 *pi

% There are a lot of experiments to be carried out.
% Vary N
% Vary sigma - too much noise and you may miss the signal subspace.
% Vary Neig, we do not always know the rank of the signal subspace
% Vary M
```