Holger Broman, August 19, 1999

Matrices, geometry, and mappings

The intuitive aspect

As was seen in the section on “Linear systems of equations, geometry, and algebra”, several interesting discussions on the properties of matrices arise from the equation

$$\mathbf{y} = A\mathbf{x}.$$

Let us study an example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} ; \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Longrightarrow$$

$$\mathbf{y} = A\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} .$$

- As you see, A maps $\mathbf{x} \in \mathbb{R}^2$ to $\mathbf{y} = A\mathbf{x} \in \mathbb{R}^3$.
- Not all of $\mathbb{R}^3$ can be reached by the mapping; the range of the mapping is the plane spanned by the first two components of $\mathbf{y}$.
- Recall that $A\mathbf{x}$ stands for a linear combination of the columns of A to make the preceding comment obvious.

This argument will be formalized later on to define the *column space* of a matrix. This space equals the *range space* of the mapping defined by the matrix.

———— o O o ————

There is something fascinating about the zero in the third position of $\mathbf{y}$. We will expand on that by studying a second example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} ; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Longrightarrow$$

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, solve the equation:

$$A\mathbf{x} = 0$$

$$\Longrightarrow$$

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$$

This means that there is an infinity of vectors $\mathbf{x}$ that produce the output 0 when mapped by A. This argument will be formalized later on to define the *null space* of a matrix.

- Note that the range space of A is the entire space (plane) onto which A maps.

Exercise: To illustrate range and null spaces "in one shot", study:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix A happens to be a projection matrix. We will speak more of projections later on.

—————— o O o ——————

There are many things taken for granted so far. For instance, the concepts of vectors and matrices, addition of vectors, scalar products and so on. A few such basic concepts will be defined further on. For now, arguing mainly from an intuitive point of view, the following recommendation may be helpful in becoming friends with abstract linear algebra.

Use your familiarity with 2D and 3D to understand the formalism, use the formalism to become familiar with N D.

Examples:

- Two orthogonal vectors, $\mathbf{x}$ and $\mathbf{y}$, in the plane

  $\Longleftrightarrow$

  $\mathbf{x}^T\mathbf{y} = 0$

  - - ->

  the concept of orthogonal vectors in $\mathbb{R}^N$.

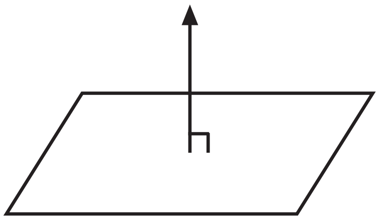
- The norm (length) of a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in 2D is (Pythagoras):

  $\| \begin{pmatrix} a \\ b \end{pmatrix} \| = (a^2 + b^2)^{1/2}$

  - - ->

  the norm of an N-dimensional vector.

- A normal vector to a plane is orthogonal to all vectors in the plane

  The real line (the normal) is a 1D subspace of 3D, the plane a 2D subspace.

  - - ->

  Construct the abstract concept of two orthogonal spaces of dimensions N and M, respectively:

  > Let $\mathbf{x} \in \mathbb{R}^{N+M}$. Then all vectors in one subspace are orthogonal to all vectors in the other.

- Consider a plane in 3D. The "rest" of 3D is of course the line spanned by a normal to the plane, i.e. a vector orthogonal to all vectors in the plane. Expand this idea to construct the complementary space to an M-dimensional subspace of $\mathbb{R}^N$.

In conclusion, linear mappings from $\mathbb{R}^N$ to $\mathbb{R}^M$ and $M \times N$ matrices may be considered as the same thing: A linear mapping corresponds uniquely to a matrix (and vice versa).

Please run the m-file rangespace in Matlab.

```
% rangespace.m, illustration of range-space
% run this program several times - random data.
clear

% Generate a special matrix, what is special?
A=[1 0 0 ; 1 1 0 ; 0 1 0];
A(:,1)=randn(1)*A(:,1);
A(:,2)=randn(1)*A(:,2);
A(:,3)=randn(1)*A(:,1)+randn(1)*A(:,2);

% Generate 50 random vectors
x=randn(3,50);

% Exemplify the range-space of A by plotting the 'output'.
y=A*x;
figure(1), clf, hold on, axis equal, view(3)
plot3(y(1,:),y(2,:),y(3,:),'*')

% Plot a frame
n=ceil(max(max(abs(y))));
y(:,1)=[-n -2*n -n]';
y(:,2)=[-n 0 n]';
y(:,3)=[n 2*n n]';
y(:,4)=[n 0 -n]';
y(:,5)=[-n -2*n -n]';
plot3(y(1,1:5),y(2,1:5),y(3,1:5),'y')

% Plot the column vectors of A - you may need to remove the plotting of
% the 50 points to see what happens.
plot3([0 A(1,1)],[0 A(2,1)],[0 A(3,1)],'r')
plot3(A(1,1),A(2,1),A(3,1),'pr')
plot3([0 A(1,2)],[0 A(2,2)],[0 A(3,2)],'r')
plot3(A(1,2),A(2,2),A(3,2),'pr')
plot3([0 A(1,3)],[0 A(2,3)],[0 A(3,3)],'r')
plot3(A(1,3),A(2,3),A(3,3),'pr')

% try various viewpoints by specifying, for instance,
% view(-69.5,30)
% view(3) to return to default, see help view
```