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Matrices, geometry, and mappings

Vector spaces, linear mappings, independence, and bases

Now we know intuitively what a vector is, and what an N-dimensional space is. There is a need, however, to know more firmly, i.e. formally.

> **Definition:** A vector space is a set of elements (we usually call them vectors) together with rules for addition of elements and multiplication of an element by a scalar. The result of adding two elements or multiplying an element by a scalar must be an element that is a member of the original set.

Note 1: The addition and multiplication must obey the commutative and distributive laws, for example

$$\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$$

$$c(\mathbf{x}_1 + \mathbf{x}_2) = c\,\mathbf{x}_1 + c\,\mathbf{x}_2$$

Note 2: A vector space is also called a linear space.

Note 3: Of course, $\mathbb{R}^N$ is a vector space.

Note 4: The concept of a vector space is, however, much wider than that. For instance, the set of all mappings from $\mathbb{R}^1$ to $\mathbb{R}^1$ with conventional addition is a vector space.

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We need to know what a linear mapping is:

> **Definition:** A mapping, M, is linear iff
>
> $$M(a_1 + a_2) = M(a_1) + M(a_2)$$
>
> $$M(c\,a) = c\,M(a) \qquad c \in \mathbb{R}^1\,(\mathbb{C}^1)$$

Note 5: Linear mappings obey the *superposition principle*, which is expressed in this definition.

Exercise 1: Given the mapping from $x \in \mathbb{R}^1$ to $y \in \mathbb{R}^1$

$$y = k\,x + \ell,$$

where the 'parameters' $k, \ell \in \mathbb{R}^1$, which values of k and ℓ make the mapping linear?

Note 6: An M by N matrix A represents a linear mapping from $\mathbb{R}^N$ to $\mathbb{R}^M$:

$$\mathbf{y} = A\,\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^N, \qquad y \in \mathbb{R}^M$$

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We will now define linear independence and the concept of a basis for a vector space.

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_K$ are linearly independent iff $$\sum_{k=1}^{K} c_k \mathbf{v}_k = 0 \quad \Longrightarrow \quad c_k = 0 \ \forall\ k$$

Note 7: Manipulate the equation above:

$$0 = \sum_{k=1}^{K} c_k \mathbf{v}_k = [\mathbf{v}_1 \ldots \mathbf{v}_K] \begin{bmatrix} c_1 \\ \vdots \\ c_K \end{bmatrix} = V\,\mathbf{c}$$

An equivalent statement to make is thus:

The column vectors of the matrix V are linearly independent iff $$V\,\mathbf{c} = 0 \quad \Longrightarrow \quad \mathbf{c} = 0$$

Note 8: In $\mathbb{R}^N$, at most N vectors can be linearly independent. We abstain from proving this.

Now let us study vectors that span a vector space and define the concept of a basis for the space.

The N-dimensional vectors $\mathbf{v}_1, \ldots, \mathbf{v}_K$ span $\mathbb{R}^N$ iff for every $\mathbf{x} \in \mathbb{R}^N$ you can find a set of coefficients c_k so that

$$\mathbf{x} = \sum_{k=1}^{K} c_k \, \mathbf{v}_k$$

Note 9: The vectors $\{\mathbf{v}_k\}$ span the space if you can reach any point in the space by means of some linear combination.

Note 10: The above can be written

$$\mathbf{x} = V\,\mathbf{c}, \qquad V = [\mathbf{v}_1, \ldots, \mathbf{v}_K]\,, \qquad \mathbf{c}^T = [c_1, \ldots, c_K]$$

Note 11: It is required that $K \geq N$

Vectors $\{\mathbf{v}_k\}$ that

- span $\mathbb{R}^N$
- are linearly independent

are said to form a basis for $\mathbb{R}^N$.

Note 12: A basis is never unique. Consider, for instance,
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on one hand, and
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ on the other. Both sets of vectors are linearly independent and span the plane.

Note 13: The concept of dimension of a space originates *not* from the length of the vectors but from the number of vectors in a basis for the space.

Example: The vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

span a plane in $\mathbb{R}^3$. The plane has dimension 2.

An orthogonal basis is a basis where all vectors are mutually orthogonal.

An orthonormal basis is an orthogonal basis where all vectors have unit length.

If N linearly independent vectors span a space, then any other set of vectors that are linearly independent and span the same space must have N members. This is the fundamental property embodied in the concept of dimension.

Please run the m-file bases2D in Matlab.

```
% bases2D.m
% run this program several times, random data.
clear

% get two random vectors
v1=3*rand(2,1);
v2=3*rand(2,1);

% Make them orthogonal
v2o=v2-v1'*v2*v1/(v1'*v1);

% Make them orthonormal
v1on=v1/norm(v1);
v2on=v2o/norm(v2o);

% Get suitable scaling for plots
k=max([norm(v1) norm(v2) norm(v2o) 1]);
ax=[-k k -k k];

% Plot the stuff
figure(1), clf, axis(ax), axis equal, hold on
plot([0 v1(1)], [0 v1(2)], 'r', [0 v2(1)], [0 v2(2)], 'g')
plot(v1(1), v1(2), 'rp', v2(1), v2(2), 'gp')
title( 'Two random vectors - a basis for the plane')

figure(2), clf, axis(ax), axis equal, hold on
plot([0 v1(1)], [0 v1(2)], 'r', [0 v2o(1)], [0 v2o(2)], 'g')
plot(v1(1), v1(2), 'rp', v2o(1), v2o(2), 'gp')
title( 'The vectors made orthogonal - an orthogonal basis')

figure(3), clf, axis(ax), axis equal, hold on
plot([0 v1on(1)], [0 v1on(2)], 'r', [0 v2on(1)], [0 v2on(2)], 'g')
plot(v1on(1), v1on(2), 'rp', v2on(1), v2on(2), 'gp')
title(' The vectors made orthonormal - an orthonormal basis')
```