Holger Broman, August 19, 1999

Application Example: MUSIC

The following example is a version of an algorithm, MUSIC, with reference to problems in communication. This is the scenario: we have a scalar signal, $y(t)$, which is the sum of L complex sinusoids in additive noise,

$$y(t) = \sum_{\ell=1}^{L} b_\ell \, e^{j\omega_\ell t} + e(t).$$

The task at hand is to estimate the frequencies ω_ℓ from the observations $y(t)$. We start by stacking some observations in a vector:

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-M) \end{bmatrix} = \text{ use the model of } y(t) =$$

$$= \begin{bmatrix} 1 & & 1 \\ e^{-j\omega_1} & & e^{-j\omega_L} \\ \vdots & \cdots & \vdots \\ \vdots & & \vdots \\ e^{-jM\omega_1} & & e^{-jM\omega_L} \end{bmatrix} \begin{bmatrix} b_1 \, e^{j\omega_1 t} \\ \vdots \\ \vdots \\ \vdots \\ b_L \, e^{j\omega_L t} \end{bmatrix} + \begin{bmatrix} e(t) \\ \vdots \\ \vdots \\ \vdots \\ e(t-M) \end{bmatrix}$$

We note with pleasure that the matrix involved is vandermonde, and thus has full rank for ω_ℓ distinct. Introduce the notations

$$\mathbf{a}(\omega) = \begin{bmatrix} 1 & e^{-j\omega} & \cdots & e^{-jM\omega} \end{bmatrix}^H ,$$

and

$$s_\ell = b_\ell \, e^{j\omega_\ell t},$$

so that

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{a}(\omega_1) & \cdots & \mathbf{a}(\omega_L) \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_L \end{bmatrix} + \text{ noise} = A\,\mathbf{s} + \text{ noise.}$$

Now, we make some observations:

- The steering vector $\mathbf{a}(\omega)$ is a curve in C^{M+1}.
- For $M \geq L-1$, $\{\mathbf{a}(\omega_\ell)\}_{\ell=1}^{L}$ span an L-dimensional subspace of C^{M+1}, the signal subspace.

- The set $\{\omega_\ell\}_{\ell=1}^{L}$ is the solution to the intersection of $\mathbf{a}(\omega)$ and the signal subspace. The set is unique as the vandermonde matrix has full rank equal to L.

In conclusion, it seems to be a good idea to estimate the signal subspace. One way to do this is to estimate the correlation matrix

$$\hat{R} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{y}(n)\, \mathbf{y}^H(n).$$

As this is a course on linear algebra, we disregard the influence of the noise.

$$\begin{aligned} \hat{R} =& \frac{1}{N} \sum_{n=1}^{N} A\, \mathbf{s}(n)\, \mathbf{s}^H(n)\, A^H = \\ =& A \left[\frac{1}{N} \sum_{1}^{N} \mathbf{s}(n)\, \mathbf{s}^H(n) \right] A^H = A\, \hat{R}_{\mathbf{s}}\, A^H \end{aligned}$$

provided that $\hat{R}_{\mathbf{s}}$ is full rank.

Hurray, the range-space of $\hat{R}$ equals the range-space of A, equals the signal subspace.

Note 1: When noise is present, assume that it is white and small. Then take the L largest eigenvalues of $\hat{R}$ — $\hat{R}$ Hermitian implies real-valued non-negative eigenvalues. The corresponding eigenvectors span the estimate of the signal subspace.

Let S be the matrix of dimensions $(M+1) \times L$ that has orthonormal columns that span the signal subspace. As the norm of $\mathbf{a}(\omega)$ equals $M+1$, independent of ω, the desired solution is found by finding the L maxima to the following:

$$\max_{\omega} \left| \mathbf{a}^H(\omega)\ S \right|^2$$

To produce nice plots, you can construct the MUSIC pseudo-spectrum

$$P(\omega) = \frac{1}{1 - \frac{\left|\mathbf{a}^H(\omega)\ S\right|^2}{\left|\mathbf{a}(\omega)\right|^2}}$$

Application Example: ESPRIT

Refer back to the example presenting MUSIC to find the relevant model for the data

$$\mathbf{y}(t) = A\,\mathbf{s}(t) + e(t),$$

where the steering matrix A is

$$A(\omega) = [\mathbf{a}(\omega_1) \ \cdots \ \mathbf{a}(\omega_L)] =$$

$$= \begin{bmatrix} 1 & 1 \\ e^{-j\omega_1} & e^{-j\omega_L} \\ \vdots & \vdots \\ \vdots & \vdots \\ e^{-jM\omega_1} & e^{-jM\omega_L} \end{bmatrix}.$$

Now, partition A in two different ways:

$$A = \begin{bmatrix} A_1 \\ \text{last row} \end{bmatrix} = \begin{bmatrix} \text{first row} \\ A_2 \end{bmatrix}.$$

It follows

$$A_2 = A_1\,\Phi,$$

where

$$\text{diag}(\Phi) = \left(e^{-j\omega_1}, \ldots, e^{-j\omega_L}\right).$$

So, to estimate the frequencies, find Φ.

Unfortunately, we do not have A. We can estimate, however, the signal subspace from the data, and construct the matrix S that spans the same subspace as does A, as demonstrated in the derivation of the MUSIC algorithm. This means that there exists a square full rank transformation matrix that relates A and S:

$$S = A\,T.$$

Now partition S as we did with A. Then

$$S_1 = A_1\,T$$

$$S_2 = A_2\,T,$$

and

$$A_2 = A_1\,\Phi$$

implies

$$S_2\,T^{-1} = S_1\,T^{-1}\,\Phi,$$

which in turn leads to

$$S_2 = S_1\,T^{-1}\,\Phi\,T = S_1\,\Psi.$$

As Ψ and Φ are related by a similarity transform, they have the same eigenvalues, and we find the frequencies by finding the eigenvalues of Ψ. In practice, you need to solve

$$S_2 = S_1 \Psi$$

in a least squares sense. In Matlab code, it is really simple:

$$\Psi = S_1 \backslash S_2$$

will do it.