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Permutations

Permutation matrices occur now and then in signal processing problems. One example can be found in the blind source separation scenario. The situation is as follows: you have N signal sources that are mixed in a linear MIMO system (Multiple Input Multiple Output):

Now you desire to de-mix the signals y_n, and you use a MIMO de-mixer:

You design the de-mixer so that the outputs s_n each depend on one source signal only! How do you ascertain that s_1 depends on x_1 and so on? The answer is that you cannot always do that, but you can still separate the signals so that

$$s_n = x_k \quad \text{for some } k,$$

Thus, the outputs s_n are permutated versions of the inputs x_n.

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An N by N permutation matrix has N entries that are unity, the rest are zero. Its determinant has unit magnitude.

It follows that each row – and column – has exactly one entry which is unity. It also follows that there are $N!$ distinct permutation matrices of size N by N. Let

$$P = [\mathbf{p}_1 \cdots \mathbf{p}_N],$$

where each column vector has zeros except in one, unique position. Study

$$P^T P = \begin{bmatrix} \mathbf{p}_1^T \\ \vdots \\ \mathbf{p}_N^T \end{bmatrix} [\mathbf{p}_1 \cdots \mathbf{p}_N] = \begin{bmatrix} 1 & & & \\ & \ddots & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix} = I.$$

Thus, permutation matrices are norm preserving, and all eigenvalues fall on the unit circle.

The product of two permutation matrices is a permutation matrix. This follows by considering

$$P_1 = [\mathbf{p}_1 \cdots \mathbf{p}_N],$$

and

$$P_2 = [\mathbf{p}_{n1} \cdots \mathbf{p}_{nN}],$$

where P_2 has scrambled the columns of P_1. Now, each row of P_1 has one unity entry that is guaranteed to hit the unity entry of exactly one column of P_2 so that

$$P_1 P_2 = P_3,$$

where P_3 is a permutation matrix.

It seems reasonable that if you keep on permuting the entries of a vector with the same permutation matrix, you will sooner or later return to the original vector. This means that there exists a positive integer K so that

$$P^K \mathbf{x} = \mathbf{x},$$

and that

$$P^K = I.$$

This is indeed true, but we do not give the proof here.

Exercise: List all six permutation matrices of size 3×3 and show the above relationship.

Note: A consequence of $P^K = I$ is that the eigenvalues to P solve the equation $\lambda^K = 1$.

Please run the m-file permutations in Matlab.

```
% permutations.m
clear

% Generate a random permutation matrix
N=5;
v=rand(5,1);
[u,k]=sort(v);
P=zeros(N,N);
for n=1:N
P(n,k(n))=1;
end

% Test what P does to a matrix
A=zeros(N,N);
B=zeros(N,N);
for n=1:N
A(n,:)=(n-1)*N+1:n*N;
B(:,n)=((n-1)*N+1:n*N)';
end

sprintf('Left-multiplication by a permutation matrix equals a row-exchange')
P, A, P*A
pause

sprintf('Right-multiplication by a permutation matrix equals a column-exchange')
B, P, B*P
pause

% Find the periodicity of P
PP=P;
n=1;
while max(max(PP-eye(N))) > 0
n=n+1;
PP=P*PP;
end
sprintf('Power of P that equals the identity')
P, n, P^n
% Can you find the pattern in P that determines the periodicity?
```