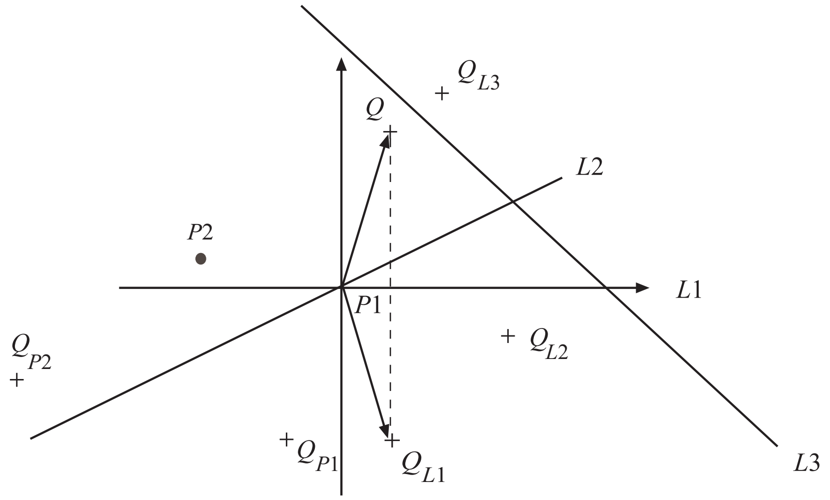
Holger Broman, August 19, 1999

Reflections

Let us start by studying an example. Given a point Q in the plane, reflect Q in some subspaces (lines and points in the plane). One example is shown explicitly.



We note that it seems as if Q, Q_{L1}, Q_{L2} and Q_{P1} fall on a circle (easily proven using elementary methods from geometry) with its center at the origin. Also note that Q_{L3} and Q_{P2} do <u>not</u> fall on the same circle.

We also note that $L1$, $L2$ and $P1$ are all linear subspaces (contain the origin), whereas $L3$ and $P2$ are not.

The global conclusion is:

Reflections in linear subspaces are norm-preserving.

You can also easily convince yourself that an orthogonal reflection in linear subspaces is a linear mapping – use R^2, select two points Q_1 and Q_2, and let your intuition work. Thus, there must exist something worthy of the name reflection matrix.

The last piece of evidence we need to make a sensible definition of a reflection matrix is to note that the mirror image of a mirror image takes us back to where we started.

First step towards the definition: Denote the vector to the point Q by $\mathbf{x}$, and the reflection matrix by M. Then

$$\| M\mathbf{x} \| = \| \mathbf{x} \|$$

and

$$M^2\mathbf{x} = \mathbf{x}.$$

The first equation implies (see Norm-preserving linear mappings) that

$$M^T M = I,$$

the second that

$$MM = I,$$

so that M is symmetric. We can now make the sensible definition:

Definition: A symmetric N by N matrix M is called a reflection matrix iff M^2 equals the identity matrix.

Note 1 We have used "reflection" as short for "orthogonal reflection". There exist strange things called oblique reflections.

Note 2 Can you see how reflection can be used to highlight the concept of projection?

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The eigenvalues of reflection matrices are easily found:

$$M\mathbf{g} \triangleq \lambda\mathbf{g}$$

$$\mathbf{g} = I \cdot \mathbf{g} = M^2\mathbf{g} = M\lambda\mathbf{g} = \lambda M\mathbf{g} = \lambda^2\mathbf{g}$$

Thus

$$\lambda \in \{-1, 1\}$$

Note 3 Can you guess how to determine the number of eigenvalues that equal 1?

Hint: Let us see what you make of the following examples.

Example 1 Which matrix produces Q_{L1} in the figure on page 1?

Example 2 Which matrix produces Q_{P1} in the figure on page 1?

Example 3 $M_1 = -I$. What is the subspace, in particular its dimension?

Example 4 $M_2 = I$. What is the subspace, in particular its dimension?

Example 5

$$M_3 = \begin{bmatrix} 1 & & & & \\ & -1 & & 0 & \\ & & -1 & & \\ & 0 & & \ddots & \\ & & & & -1 \end{bmatrix}. \text{ What is the subspace?}$$

Note 4 This argument/result will reappear in the context of projections.

Note 5 How do you construct the reflection matrix that reflects in a given subspace? This will be shown in the section on projections.

Exercise: Show that the following matrix

$$M = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

performs a reflection in the line through the origin with angle α. The matrix M is an example of a Householder reflection, also known as Householder matrix or Householder transformation.

Please run the m-file reflections in Matlab. Please note that the construction of the reflection matrix follows the procedure outlined in the leaflet on projections.

```
% reflections.m, reflections in a plane in 3D
% run this program several times, random data.
clear

v1=randn(3,1); v2=randn(3,1); A=[v1 v2];

% Construct the reflection matrix, M, and check for key properties
P=A*inv((A'*A))*A'; M=2*P-eye(3);
norm(M-M','fro'), norm(M*M-eye(3),'fro')

% Create an ON-basis for the plane and generate plane-plot
[q,r]=qr(A,0);
step=0.2; width=3; index=1;
for l=-width:step:width
p1=l*q(:,1)-width*q(:,2);
p2=l*q(:,1)+width*q(:,2);
p3=(l+step/2)*q(:,1)+width*q(:,2);
p4=(l+step/2)*q(:,1)-width*q(:,2);
planeplot(:,index)=p1;
planeplot(:,index+1)=p2;
planeplot(:,index+2)=p3;
planeplot(:,index+3)=p4;
index=index+4;
end
ax=[ -width width -width width -width width];

% Generate some random vectors and project them
m=50;
dum=randn(3,m);
dummer=M*dum;

% plot some examples
figure(1), clf, axis(ax), axis equal, view(3), hold on
plot3([0 v1(1)], [0 v1(2)], [0 v1(3)], 'b')
plot3([0 v2(1)], [0 v2(2)], [0 v2(3)], 'b')
for l=1:5
plot3(dum(1,l),dum(2,l),dum(3,l),'rp')
plot3(dummer(1,l),dummer(2,l),dummer(3,l),'gp')
plot3([dum(1,l) dummer(1,l)], [dum(2,l) dummer(2,l)], [dum(3,l) dummer(3,l)], 'y')
end
plot3(planeplot(1,:), planeplot(2,:), planeplot(3,:), 'k')
title('Five points and their reflections')
```