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Toeplitz matrices

Toeplitz matrices occur in many engineering applications. We will mention some, but first the definition:

> A matrix is Toeplitz if all elements on any NW-SE diagonal are the same.

Example 1: Here is a 3×2 Toeplitz matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}$$

Example 2: The correlation matrix (second order moment matrix) of a random vector. Let $\mathbf{x} = [x_0, \ldots, x_{N-1}]^T$ be a random vector. Then the correlation matrix R is defined by

$$R = E\left[\mathbf{x}\,\mathbf{x}^T\right]$$

Expand:

$$R = E\left\{ \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} [x_0 \cdots x_{N-1}] \right\} =$$

$$= E\left\{ \begin{bmatrix} x_0x_0 & x_ox_1 & \cdots & x_0x_{N-1} \\ x_1x_0 & x_1x_1 & & \\ \vdots & & \ddots & \\ x_{N-1}x_0 & & & x_{N-1}x_{N-1} \end{bmatrix} \right\}$$

Now, for wide-sense stationary random vectors we find

$$R = \begin{bmatrix} r(0) & r(1) & \cdots & r(N-1) \\ r(1) & r(0) & & \\ \vdots & & \ddots & \\ r(N-1) & & & r(0) \end{bmatrix},$$

where $r(\tau)$ is the autocorrelation function (second order moment function) of the stochastic sequence $x_0, x_1, \ldots, x_{N-1}$.

We note that R is a square, symmetric Toeplitz matrix. This implies real-valued, non-negative eigenvalues. A basic fact behind this property is that the matrix $\mathbf{x}\,\mathbf{x}^T$ has only non-negative eigenvalues.

Example 3: Linear convolutions.
Let x_0 be the input to an LTI system (Linear Time Invariant) with impulse response h_n of length K. The output is then given by the convolution sum

$$y_n = \sum_{k=0}^{K-1} h_k\, x_{n-k}$$

Now, stack $y_L, \ldots, y_M$ $(M > L)$ in a vector, use the convolution sum to find

$$\mathbf{y} = \begin{bmatrix} y_L \\ \vdots \\ \vdots \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} x_L & x_{L-1} & \ldots & x_{L-K+1} \\ x_{L+1} & x_L & \ldots & x_{L-K+2} \\ x_{L+2} & x_{L+1} & & \\ \vdots & & & \\ \vdots & & & \\ x_M & x_{M-1} & & \end{bmatrix} \begin{bmatrix} h_o \\ \vdots \\ \vdots \\ h_{K-1} \end{bmatrix} = X\,\mathbf{h}$$

The "data matrix" X is an $(M - L + 1) \times K$ Toeplitz matrix.

Example 4: Circular convolutions.
Assume that input data x_n is available for $n = 0, \ldots, N-1$ only and that the impulse response is not longer $(K \leq N)$ and that you want to calculate the output y_n for $n = 0, \ldots, N-1$. As you can see from

$$y_n = \sum_{k=0}^{N-1} h_k\, x_{n-k},$$

you will use data x for negative indices, data which is not available. One solution is of course to extend data by zeros. Another is to let the available data "wrap around", i.e. to be repeated periodically, i.e. interpret x_{n-k} for $n - k < 0$ as x_{n-k+N}. We then get (check please):

$$\mathbf{y} = \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} x_0 & x_{N-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & & & \\ x_{N-1} & \ldots & \ldots & x_0 \end{bmatrix} \begin{bmatrix} h_0 \\ \vdots \\ \vdots \\ h_{N-1} \end{bmatrix} = X\,\mathbf{h}$$

The "data matrix" X is an example of a circulant Toeplitz matrix. They are constructed by shifting the first row one step to the right and letting the last entry become the first (circular shift) in the next row. The process is repeated until the matrix is square.

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Circulant Toeplitz matrices have an interesting property. The Fourier matrix can be used to make them diagonal! Let us start with a diagonal matrix D and study FDF^H:

$$D = \mathrm{diag}\,(d_0 \cdots d_{N-1})$$

$$
\sqrt{N}\,F = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & a^1 & \cdots & a^{(N-1)} \\ \vdots & \vdots & \ddots & \\ 1 & a^{(N-1)} & & a^{(N-1)(N-1)} \end{bmatrix}, \qquad a = e^{-j\frac{2\pi}{N}}
$$

Then,

$$
\sqrt{N}\,DF^H = \begin{bmatrix} d_0 & \cdots & \cdots & d_0 \\ d_1 & d_1 a^{-1} & \cdots & d_1 a^{-(N-1)} \\ \vdots & & & \\ \vdots & & & \\ d_{N-1} & & & \end{bmatrix}
$$

and

$$
NFDF^H = \begin{bmatrix} \sum d_n & \sum d_n a^{-n} & \cdots & \sum d_n a^{-n(N-1)} \\ \sum d_n a^n & \sum d_n & \ddots & \\ \vdots & \ddots & & \\ \sum d_n a^{n(N-1)} & & & \sum d_n \end{bmatrix},
$$

which is Toeplitz, circular shift. Introduce the notation

$$
C_T = FDF^H
$$

for the circulant Toeplitz matrix. Then — remember $F^H = F^{-1}$

$$
F^H C_T F = D,
$$

so that

$$
\boxed{FAF^H \text{ is diagonal iff } A \text{ is a circulant Toeplitz matrix.}}
$$

Note. Obviously, C_T^* is also circulant Toeplitz. This implies (show it) that $F^H AF$ is diagonal iff A is circulant Toeplitz.

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The eigenvalues of a circulant Toeplitz matrix are easy to find. Just do the following:

$$
\begin{aligned}
\det(C_T - \lambda I) &= \det(FF^H)\det(C_T - \lambda I) = \\
&= \det\left[F\left(C_T - \lambda I\right)F^H\right] = \det\left(FC_TF^H - \lambda I\right)
\end{aligned}
$$

As FC_TF^H is diagonal and has the same eigenvalues as has C_T, the eigenvalues of C_T can be read from the diagonal elements.

The diagonalization of a circulant Toeplitz matrix by the Fourier matrix mirrors the connection between convolution and multiplication through the Fourier transform.

You should run the m-file fourmat in Matlab when you have studied the leaflet on Fourier, Toeplitz and Hankel matrices.