MATEMATIK
Chalmers

2009-08-26

Tentamen i Inledande matematik för E, Z och V/AT

1. Till denna uppgift ska du **endast lämna in svar**, alltså utan motiveringar.

   a) Bestäm alla lösningar till ekvationssystemet (2p)

$$\begin{cases} x - y + 2z = 0 \\ x + y - 2z = 2 \end{cases}$$

   b) Ge det komplexa talet $(-1+i)^{20}$ på formen $a+bi$. (2p)

   c) För vilka reella tal $x$ är $\dfrac{|x+2|}{x-2} \geq 0$? (2p)

   d) Bestäm största och minsta värdet av funktionen $f(x) = \dfrac{2+\sin x}{3-2\sin x}$. (2p)

   e) Beräkna följande gränsvärden: (3p)

   i. $\displaystyle\lim_{x\to 0} \frac{\ln(1+2x)}{\sin x}$ ii. $\displaystyle\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{3x}$ iii. $\displaystyle\lim_{x\to\infty} \frac{x^5\, 2^x}{(x^4+x^2)3^x}$

   f) Funktionen $y(x)$ definieras av ekvationen $y^3 + y = x$. (3p)
   Beräkna $y(2)$, $y'(2)$ och $y''(2)$.

**Till uppgifterna 2-5 ska du lämna in fullständiga lösningar**.

2. a) Ange en ekvation för det plan som är parallellt med planet $x + 2y + 3z = 1$ och innehåller punkten $(1,2,3)$. (2p)

   b) Visa att planen $x - z = 3$ och $x + y + z = 1$ skär varandra och bestäm en riktningsvektor för deras skärningslinje. (2p)

   c) Finn en ekvation för det plan som innehåller skärningslinjen mellan planen $x + y - 2z = 6$ och $2x - y + z = 2$ och har normalvektorn $(-3,3,-4)$. (2p)

**Var god vänd!**

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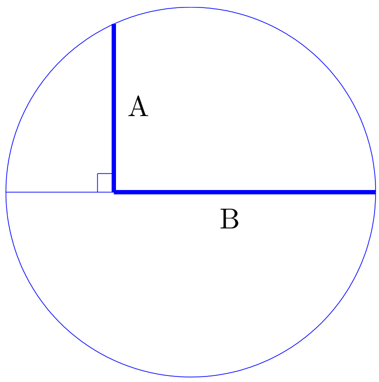
3. Ange definitions- och värdemängden till funktionen (6p)

$$f(x) = \sqrt{4-x} + \sqrt{2x-2}.$$

4. Rita grafen till funktionen $f(x) = 12x^5 - 15x^4 - 20x^3 + 30x^2 - 15$. (6p)
Ange eventuella lokala extrempunkter. Visa också att grafen saknar asymptoter. (Konvexitet/konkavitet behöver inte utredas.)

5. En rät vinkel kan flyttas längs diametern i en cirkel av radie $r$, se figuren nedan. Vad är den största möjliga längden $A + B$ som skärs av av cirkeln ? (6p)




6. Avgör vilka av följande påståenden som är sanna respektive falska. Du behöver inte motivera dig. Rätt svar ger 1p, inget svar 0p och fel svar -1p. Dock ej mindre än 0p totalt. (6p)

   a) Om en funktion inte är kontinuerlig, så är den inte heller deriverbar.
   b) Funktionen $f(x) = x^3 + x + 2$ är inverterbar.
   c) Om $\mathbf{u}$ och $\mathbf{v}$ är två vektorer i rummet, så måste vektorn $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$ vara vinkelrät mot vektorn $\mathbf{v}$.
   d) $\arctan(\tan x) = x$ för alla $x$ i definitionsmängden för $\tan x$.
   e) $\tan(\arctan x) = x$ för alla $x$ i definitionsmängden för $\arctan x$.
   f) Om $|f(x)|$ är deriverbar i $x = 0$, så måste också $f(x)$ vara deriverbar i $x = 0$.

7. a) Definiera *derivatan* av en funktion $f$ i en punkt $x$. (2p)
   b) Formulera och bevisa *produktregeln*, dvs räknelagen för derivering av en produkt av två funktioner. (4p)

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**Lösningar**

1.(a) In matrix form the system is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 1 & -2 & 2 \end{array}\right).$$

A sequence of row operations transforms the system to the echelon form

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right).$$

Thus we can choose $z = t$ as a free variable. Back substitution gives, first of all,

$$y - 2z = 1 \Rightarrow y = 1 + 2t,$$

and then

$$x - y + 2z = 0 \Rightarrow x - (1 + 2z) + 2z = 0 \Rightarrow x = 1.$$

ANSWER : $\{(x, y, z) = (1, 1 + 2t, t) : t \in \mathbb{R}\}$.

(b) In polar form,

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$

and hence, by De Moivre's theorem,

$$(-1 + i)^{20} = 2^{10}(\cos 15\pi + i\sin 15\pi) = 2^{10}(-1) = -2^{10}.$$

(c) The numerator, being an absolute value, is always non-negative, and zero at $x = -2$. Hence, the quotient is positive if and only if the denominator is so, and this is the case when $x > 2$. Hence the quotient is non-negative if $x = -2$ or $x > 2$.
ANSWER : $x \in \{-2\} \cup [2, \infty)$.

d When $\sin x$ attains its maximum (resp. minimum) value, then the numerator attains its maximum (resp. minimum) value and, simoultaneously, the denominator attains its minimum (resp. maximum) value. Hence the quotient also attains its maximum (resp. minimum) value here. Since the maximum (resp. minimum) value of $\sin x$ is $+1$ (resp. $-1$), the maximum (resp. minimum) value of $f$ is $\frac{2+1}{3-2} = 3$ (resp. $\frac{2-1}{3+2} = \frac{1}{5}$.
ANSWER : Max. value is 3 and min. value is 1/5.

(e) **(i)** Both the numerator and denominator go to zero as $x \to 0$. Rewrite the quotient as

$$\frac{\ln(1 + 2x)}{2x} \cdot \frac{2x}{\sin x}$$

and observe that, as $x \to 0$, these two quotients go respectively to 1 and 2. Hence the limit is 2.
**(ii)** We can write

$$\left(1 + \frac{2}{x}\right)^{3x} = \left[\left(1 + \frac{2}{x}\right)^{x/2}\right]^6.$$

The inner function tends to $e$ as $x \to \infty$, hence the whole thing tends to $e^6$.
**(iii)** Exponentials defeat polynomials as $x \to \infty$ and the denominator has the larger exponential function. Hence the quotient goes to zero.

(f) When $x = 2$ we have $y^3 + y = 2$ and one sees directly that $y = 1$. Implicit differentiation gives

$$(3y^2 + 1)y' = 1 \Rightarrow y' = \frac{1}{3y^2 + 1}.$$

At $x = 2$ we've already seen that $y = 1$ and hence $y'(2) = \frac{1}{3 \cdot 1^2 + 1} = \frac{1}{4}$.
A further implicit differentiation yields

$$y'' = \frac{-6yy'}{(3y^2 + 1)^2}.$$

At $x = 2$, we've seen that $y = 1$ and $y' = 1/4$, hence

$$y''(2) = \frac{-6 \cdot 1 \cdot \frac{1}{4}}{(3 \cdot 1^2 + 1)^2} = -\frac{3}{32}.$$

ANSWER : $y(2) = 1$, $y'(2) = 1/4$, $y''(2) = -3/32$.

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2. (a) The equation of any parallel plane has the form $x + 2y + 3z = d$. Inserting $(1, 2, 3)$ we find that $d = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 14$. Hence the plane in question is $x + 2y + 3z = 14$.

   (b) Finding the intersection between the two planes means solving the given system of two equations. Let $z = t$ be the free variable and one finds that the general solution is $x = 3 + z$, $y = 1 - x - t = 1 - (3 + t) - t = -2 - 2t$. Hence the intersection of the two planes is the set of points $(3 + t, -2 - 2t, t)$, where $t \in \mathbb{R}$. We can write this set in the parametric vector form

$$\{(3, -2, 0) + t \cdot (1, -2, 1) : t \in \mathbb{R}\}.$$

   This is the equation of a line in the direction of $(1, -2, 1)$.

   (c) The equation of a plane with normal vector $(-3, 3, -4)$ has the form $-3x + 3y - 4z = d$. To find $d$, we just need one point in the plane, in other words, just one point of intersection between the planes $x + y - 2z = 6$ and $2x - y + z = 2$. Putting $x = 0$, say, we easily find that $(0, -10, -8)$ is a point of intersection. Hence $d = -3 \cdot 0 + 3 \cdot (-10) - 4 \cdot (-8) = 2$.

   Answer : The plane is $-3x + 3y - 4z = 2$.

3. The first term is only defined for $x \leq 4$ and the second term only for $x \geq 1$, hence the domain of $f$ is the interval $[1, 4]$. At the endpoints we have $f(1) = \sqrt{3}$, $f(4) = \sqrt{6}$. To find the critical points, we differentiate :

$$f'(x) = \frac{-1}{2\sqrt{4 - x}} + \frac{2}{2\sqrt{2x - 2}}.$$

   Thus $f'(x) = 0 \Leftrightarrow 2\sqrt{4 - x} = \sqrt{2x - 2} \Leftrightarrow 4(4 - x) = 2x - 2 \Leftrightarrow x = 3$.
   We compute $f(3) = \sqrt{1} + \sqrt{4} = 3$. Now since $3 > \sqrt{6} > \sqrt{3}$, we can conlcude that this single critical point is a local maximum and that the function takes on values between $\sqrt{3}$ and 3.

   Answer : $D_f = [1, 4]$, $V_f = [\sqrt{3}, 3]$.

4. *Step 0* : Search for any obvious symmetries.
There aren't any.

*Step 1* : Investigate behaviour as $x \to \pm\infty$.
Noting that $f(x)$ is a polynomial of odd degree, we have that

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = -\infty.$$

Thus no horisontal asymptotes. Since $\frac{f(x)}{x}$ behaves like $x^4$ and tends to $\infty$ as $x \to \pm\infty$, there are no other asympotes at $\pm\infty$.

*Step 2* : Investigate the domain and possible vertical asymptotes.
There aren't any vertical asymptotes (polynomials are continuous everywhere).

*Step 3* : Find and classify the critical points.
Differentiate to get

$$f'(x) = 60x^4 - 60x^3 - 60x^2 + 60x.$$

Hence we must solve the equation

$$60x^4 - 60x^3 - 60x^2 + 60x = 0 \Rightarrow 60x(x^3 - x^2 - x + 1) = 0.$$

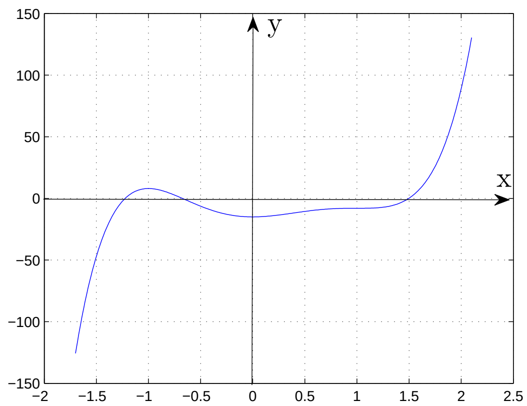
Hence $x = 0$ is one critical point. Furthermore, observe that

$$x^3 - x^2 - x + 1 = (x^3 + 1) - x(x + 1)$$
$$= (x + 1)(x^2 - x + 1) - x(x + 1) = (x + 1)(x^2 - 2x + 1) = (x + 1)(x - 1)^2.$$

Hence the other critical points are at $x = \pm 1$. Then, since

$$f(x) = 60x(x + 1)(x - 1)^2,$$

a sign table easily shows that $x = -1$ is a local maximum, $x = 0$ is a local minimum and $x = 1$ is an inflection point.



5. Let $B := (1+\alpha)r$. Then Pythagoras gives

$$(\alpha r)^2 + A^2 = r^2 \Rightarrow A = \sqrt{1-\alpha^2}\, r.$$

Thus $A + B = f(\alpha) \cdot r$, where

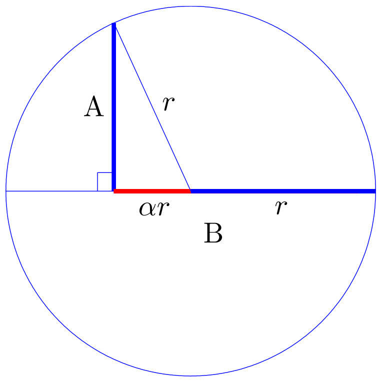
$$f(\alpha) = 1 + \alpha + \sqrt{1-\alpha^2}.$$

We seek the maximum of the function $f(\alpha)$ for $\alpha \in [-1, 1]$. We must check the endpoints and the critical points. Regarding the former, we have $f(-1) = 0$, $f(1) = 2$. Regarding the latter, we first compute

$$f'(\alpha) = 1 - \frac{\alpha}{\sqrt{1-\alpha^2}}.$$

Hence a critical point satisfies $\alpha = \sqrt{1-\alpha^2} \Rightarrow \alpha = 1/\sqrt{2}$. Finally, one easily checks that $f(1/\sqrt{2}) = 1 + \sqrt{2} > 2$, hence this is the maximum value of $f$.

Thus, the maximum value of $A + B$ is $(1+\sqrt{2})r$.



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6.(a) True. This is the contrapositive of Theorem 1 in Section 2.3 (of the seventh edition).

(b) True. It's one-to-one (since $f'(x) = 3x^2 + 1 > 0$ for all $x$) and onto (since $f$ is a polynomial of odd degree).

(c) False. The given combination will always be perpendicular to $\boldsymbol{u}$, but could lie at any angle whatsoever to $\boldsymbol{v}$.

(d) False, since the range of arctan is only the interval $(-\pi/2, \pi/2)$.

(e) True.

(f) False. For example, suppose $f(x) = -1$ for $x < 0$ and $f(x) = +1$ for $x \geq 0$. Then $f$ is not even continuous at $x = 0$, hence not differentiable either (see part (a)). But $|f(x)|$ is a constant function, hence differentiable everywhere.

7.(a)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(b) Theorem 3 in Section 2.3.