FÖ6.1 Grad, div, rot.

Först: 15.6 Flödesintegral från FÖ 5.2

16.1 Grad, div, rot.

endast sid 850 - 852 + Exempel 3, 5

Med nabla-operatorn

$$\bar{\nabla} = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$$

kan vi bilda

gradienten av skalärt fält:

$$\bar{\nabla}\phi = \operatorname{grad}\phi = \frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k}$$

och divergensen och rotationen ("curl") av vektorfält:

$$\bar{\nabla} \cdot \bar{F} = \operatorname{div} \bar{F} =$$

$$= \left(\frac{\partial}{\partial x} \bar{i} + \frac{\partial}{\partial y} \bar{j} + \frac{\partial}{\partial z} \bar{k} \right) \cdot \left(F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k} \right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\bar{\nabla} \times \bar{F} = \operatorname{rot} \bar{F} = \operatorname{curl} \bar{F}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \bar{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \bar{k}$$

<u>exempel 5</u> Stelkroppsrotation

$$\bar{v} = \Omega\left(-y\,\bar{i} + x\,\bar{j}\right)$$

$$\bar{\nabla}\cdot\bar{v} = \Omega\left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x\right) = 0$$

$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\Omega y & \Omega x & 0 \end{vmatrix} = 0\bar{i} + 0\bar{j} + 2\Omega\,\bar{k}$$

Exempel 3

$$\bar{F} = m\,\frac{\bar{r}}{r^3} = \frac{m}{r^3}\left(x\,\bar{i} + y\,\bar{j} + z\,\bar{k}\right)$$

$$\frac{\partial F_1}{\partial x} = m\,\frac{\frac{\partial x}{\partial x}\,r^3 - x\,\frac{\partial r^3}{\partial x}}{r^6} = m\,\frac{r^3 - x\cdot 3r^2\frac{\partial r}{\partial x}}{r^6}$$

$$= m\,\frac{r^3 - x\cdot 3r^2\cdot\frac{x}{r}}{r^6}$$

$$= m\,\frac{r^2 - 3x^2}{r^5}$$

På samma vis

$$\frac{\partial F_2}{\partial y} = m\,\frac{r^2 - 3y^2}{r^5}$$

$$\frac{\partial F_3}{\partial z} = m\,\frac{r^2 - 3z^2}{r^5}$$

så att

$$\bar{\nabla}\cdot\bar{F} = m\,\frac{3r^2 - 3(x^2+y^2+z^2)}{r^5} = 0$$

för $r \neq 0$.

16.2 Räkneregler

Sats 3

a) $\bar{\nabla}(\phi\psi) = \psi\bar{\nabla}\phi + \phi\bar{\nabla}\psi$

b) $\bar{\nabla}\cdot(\phi\bar{F}) = \bar{\nabla}\phi\cdot\bar{F} + \phi\bar{\nabla}\cdot\bar{F}$

c) $\bar{\nabla}\times(\phi\bar{F}) = \bar{\nabla}\phi\times\bar{F} + \phi\bar{\nabla}\times\bar{F}$

d) $\bar{\nabla}\cdot(\bar{F}\times\bar{G}) = (\bar{\nabla}\times\bar{F})\cdot\bar{G} - \bar{F}\cdot(\bar{\nabla}\times\bar{G})$

e) $\bar{\nabla}\times(\bar{F}\times\bar{G}) =$ se boken

f) $\bar{\nabla}(\bar{F}\cdot\bar{G}) =$ se boken

g) $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{F}) = 0$

h) $\bar{\nabla}\times(\bar{\nabla}\phi) = \bar{0}$

i) se boken

Bevis av a, b, c, d, g, h.

Obs: $\bar{F} = \bar{\nabla}\phi \Rightarrow \bar{\nabla} \times \bar{F} = \bar{\nabla} \times \bar{\nabla}\phi = \bar{0}$

Alltså: ett konservativt vektorfält är rotationsfritt.

Det omvända påståendet gäller i ett enkelt sammanhängande område.

Sats 4 Antag 1) $\bar{\nabla} \times \bar{F} = \bar{0}$ i D

2) D enkelt sammanhängande.

Då är $\bar{F}$ konservativt i D dvs $\bar{F} = \nabla\phi$ i D.