FÖ7.1

Värme

$\bar{\nabla}\cdot\bar{F} = f$

$\bar{F}$ flödestäthet J/m²s

u temp

$\bar{F} = -a\bar{\nabla}u$

$-\bar{\nabla}\cdot(a\bar{\nabla}u) = f$

Elasticitet

$-\bar{\nabla}\cdot\bar{\bar{\sigma}} = \bar{K}$ N/m³

$\bar{\bar{\sigma}} = \{\sigma_{ij}\}$ spänning N/m²

$\bar{u}$ förskjutning

$\bar{\bar{\varepsilon}} = \bar{\nabla}\bar{u}$ (med speciell definition av $\bar{\nabla}$)

$\bar{\bar{\sigma}} = D\,\bar{\bar{\varepsilon}}$

$-\bar{\nabla}\cdot(D\bar{\nabla}\bar{u}) = \bar{K}$

Naviers ekv.

16.? Greens formel i planet

Fundamentalsatsen:

$$\int_a^b D f(x)\,dx = \left[f(x) \right]_a^b$$

Vi ska visa en variant av detta i planet.

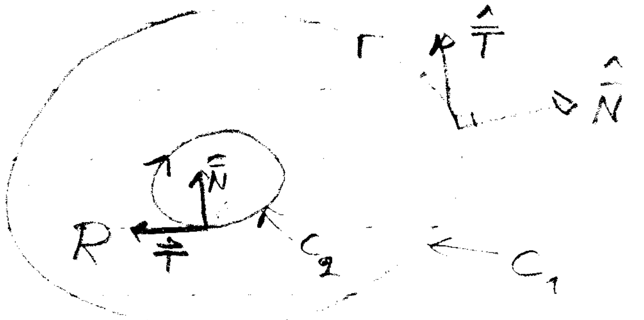
Plant vektorfält:

$$\bar{F}(x,y) = F_1(x,y)\,\bar{i} + F_2(x,y)\,\bar{j}$$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = 0\,\bar{i} + 0\,\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \bar{k}$$

Greens formel: $$\iint_R \bar{\nabla} \times \bar{F} \bullet \bar{k}\,dA = \oint_C \bar{F} \bullet d\bar{r}$$

$C = C_1 \cup C_2$



<u>positivt orienterad</u>
randkurva till R
dvs $\hat{N} \times \hat{T} = \bar{k}$, $\hat{T} = \bar{k} \times \hat{N}$

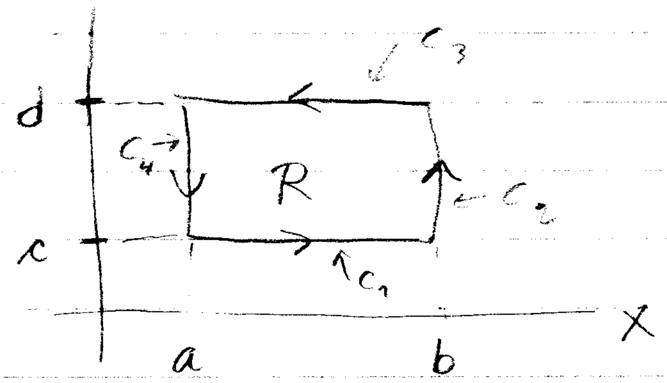
$\hat{N}$ = utåtriktade enhetsnormalen

$\hat{T}$ = enhetstangenten i kurvans riktning dvs

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \oint_C F_1\,dx + F_2\,dy$$

<u>Bevis</u> för rektangel.

R: $a \le x \le b$, $c \le y \le d$



$$-\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = -\int_a^b \int_c^d \frac{\partial F_1}{\partial y}\,dy\,dx$$

$$= -\int_a^b \Big[F_1(x,y) \Big]_{y=c}^{d} dx = -\int_a^b \big(F_1(x,d) - F_1(x,c)\big)dx$$

$$= \int_a^b F_1(x,c)\,dx - \int_a^b F_1(x,d)\,dx$$

Men: $\displaystyle \oint_C F_1(x,y)\,dx = \int_a^b F_1(x,c)\,dx - \int_a^b F_1(x,d)\,dx$

dvs

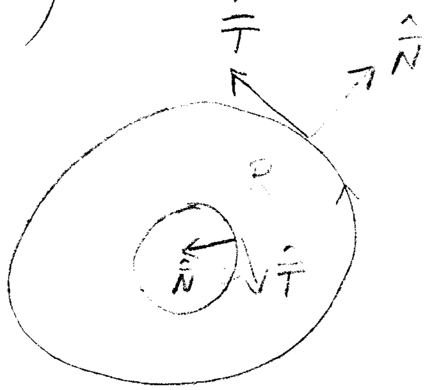
$$-\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = \oint_C F_1\,dx$$

På samma vis:

$$\iint_R \frac{\partial F_2}{\partial x}\,dx\,dy = \oint_C F_2\,dy$$

<u>Sats 7</u> (Divergenssatsen i planet)

$$\iint_R \bar\nabla\cdot\bar F\,dA = \int_C \bar F\cdot\hat N\,ds$$



<u>Bevis</u> Låt $F = F_1\bar i + F_2\bar j$ och bilda

$\bar G = -F_2\bar i + F_1\bar j$. Om $\hat N = N_1\bar i + N_2\bar j$, så $\hat T = -N_2\bar i + N_1\bar j$
$= \bar k\times\hat N$

$$\iint_R \bar\nabla\cdot\bar F\,dA = \iint_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA =$$

$$= \iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) dA = \oint_C \bar G\cdot d\bar r =$$

$$= \oint_C \bar{G} \bullet \hat{T}\, ds = \oint_C (-F_2 \bar{i} + F_1 \bar{j}) \cdot (-N_2 \bar{i} - N_1 \bar{j})\, ds$$

$$= \oint_C (F_2 N_2 + F_1 N_1)\, ds = \oint_C \bar{F} \bullet \hat{N}\, ds$$

16.4

Sats 8 (Divergenssatsen i rummet)

$$\iiint_D \bar{\nabla} \cdot \bar{F}\, dV = \iint_S \bar{F} \cdot \hat{N}\, dS$$

om D är ett område vars rand S är en orienterad yta med utåtriktad enhetsnormal $\hat{N}$.

Utan bevis.

exempel $\bar{F}(x,y,z) = m\,\dfrac{\bar{r}}{r^3}$

Vi har redan sett att $\bar{\nabla}\cdot\bar{F} = 0$ för $r>0$.

Vi beräknar flödet ut ur klotet

$x^2+y^2+z^2 \le R^2$.

Ytan är $S_R: \begin{cases} x = R\sin\phi\cos\theta \\ y = R\sin\phi\sin\theta \\ z = R\cos\phi \end{cases}$

$0\le\phi\le\pi,\quad 0\le\theta\le 2\pi$

$$\frac{\partial\bar{r}}{\partial\phi} = R\cos\phi\cos\theta\,\bar{i} + R\cos\phi\sin\theta\,\bar{j} - R\sin\phi\,\bar{k}$$

$$\frac{\partial\bar{r}}{\partial\theta} = -R\sin\phi\sin\theta\,\bar{i} + R\sin\phi\cos\theta\,\bar{j} + 0\,\bar{k}$$

$$\bar{N} = \frac{\partial\bar{r}}{\partial\phi}\times\frac{\partial\bar{r}}{\partial\theta} = R^2\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= R^2\big(\sin^2\phi\cos\theta\,\bar{i} + \sin^2\phi\sin\theta\,\bar{j} + (\sin\phi\cos\phi\cos^2\theta$$

$$+\sin\phi\cos\phi\sin^2\theta)\,\bar{k}\big)$$

$$= R^2\sin\phi\,(\sin\phi\cos\theta\,\bar{i} + \sin\phi\sin\theta\,\bar{j} + \cos\phi\,\bar{k})$$

$$= R\sin\phi\,\bar{r}$$

$$\hat{N} = \frac{\bar{N}}{|\bar{N}|} = \frac{\bar{r}}{R} = \hat{r}$$

$$dS = |\bar{N}|\, d\phi\, d\theta = R^2 \sin\phi \, d\phi \, d\theta$$

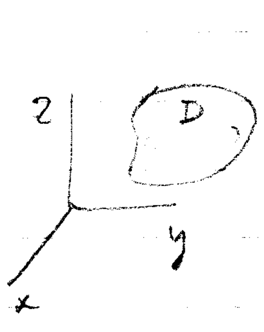
$$\iint_{S_R} \bar{F} \cdot \hat{N}\, dS = \int_0^{2\pi}\int_0^{\pi} m \frac{R\hat{r}}{R^3} \cdot \hat{r}\, R^2 \sin\phi \, d\phi \, d\theta$$

$$= m \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = m\, 2\pi \left[-\cos\phi\right]_0^{\pi} = 4\pi m$$

Låt nu $R \to 0$:

$$\lim_{R\to 0} \iint_{S_R} \bar{F}\cdot N\, dS = 4\pi m$$

Punktkälla med styrkan $4\pi m$ i origo.
Allt flöde kommer från origo.



Låt D vara ett område som inte innehåller origo.
Gauss sats ger

$$\iint_S \bar{F}\cdot\hat{N}\, dS = \iiint_D \underbrace{\nabla\cdot\bar{F}}_{=0}\, dV = 0.$$