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Additional material for the course

"Computational Geometry" MAI520

(a very preliminary version)

The course "Computational Geometry" based on O'Rourke's book "Computational Geometry in C" is/was offered for the first time at Chalmers/GU during the late fall 2000. The following notes cover the additional material that was discussed in class. However some parts of the text has not been presented there. They stem from a joint project, together with Björn Dahlberg, to collect the major ideas from the different fields of geometry – differential geometry, discrete geometry, computational geometry – and to present that material in a unified way, all aspects being present at the same time. The death of Dahlberg ruined the base for such a project and parts of these notes are some scattered debris from this undertaking. Needless to say, all shortcomings regarding these notes, are due to me.

Handouts

- V.Chvatal: A Combinatorial Theorem in Plane Geometry, Journal of Combinatorial Theory (B) 18 (1975) 39 – 41
- N.Dyn/I.Goren/S.Rippa: Transforming triangulations in polygonal domains, Computer Aided Geomtric Design **10** (1993) 531 – 536
- S.Rippa: Minimal Roughness Property of the Delaunay Triangulation, Computer Aided Geometric Design 7 (1990) 489 – 497
- D.G.Kirkpatrik/R.Seidel: The Ultimate Planar Convex Hull Algorithm?: SIAM J.Comput. 15 (1986) 287 – 299

Homework

- Exercises 1
- Exercises 2
- Exercises 3

Handbooks

- J.Goodman/J.O'Rourke (eds): Handbook of discrete and computational geometry, CRC Press 1997
- K.Rosen (ed): Handbook of discrete and combinatorial mathematics, CRC Press 2000
- J.-R.Sack/J.Urrutia (eds): Handbook of computational geometry, Elsevier 2000

Contents

1 Polygon Triangulation/Polygon Partitioning **4**
1.1 Art Gallery Theorems 4
1.2 Sorting/Order Statistics 7

2 Triangulations **12**
2.1 General Properties 12
2.2 Transformations of Triangulations 14

3 Convex Hulls **17**
3.1 Kirkpatrik-Seidel's Ultimate Convex Hull Algorithm 17

4 Delaunay Triangulations/Voronoi Diagrams **20**
4.1 Delaunay Triangulation: Existence and Uniqueness 20
4.2 Delaunay Triangulation from a Stereographic Transformation . 24
4.3 Delaunay Triangulations minimize the Dirichlet Integral . . . 25
4.4 Farthest Delaunay Triangulations 35
4.5 General Properties of Voronoi Diagrams 37
4.6 Classical Voronoi Diagrams in the Euclidean Plane 38
4.7 Pattern Recognition and Subgraphs of Delaunay Triangulations 41

5 Arrangements **48**
5.1 Duality 48
5.2 Davenport-Schinzel Sequences 50

6 Appendix: Dahlberg's Four Vertex Theorem (discrete version) **54**
6.1 Curvature and Circle of Curvature 54
6.2 Circles of Curvature for Convex Polygons 60
6.3 Cauchy's Theorem on Convex Polygons 64
6.4 Discrete Four Vertex Theorem 69

1 Polygon Triangulation/Polygon Partitioning

At a conference in Stanford in 1973 Klee posed the problem of determining the minimum number of guards sufficient to cover the interior of an n–wall art gallery room. This problem was given in a response to a request from Chvatal for an interesting challenge. Chvatal soon supplied what has become known as Chvatal's Art Gallery Theorem, i.e. $\lfloor \frac{n}{3} \rfloor$ *guards are occasionally necessary and always sufficient to cover a polygon of* n *vertices.*

1.1 Art Gallery Theorems

The first proof of Chvátal's Art Gallery Theorem was of course given by Chvátal in 1975, see [14]. His proof starts with a triangulation of the polygon, as does Fisk's, but does not use graph coloring. Rather the theorem is proven directly by induction. Although Chvátal's proof is not as concise as Fisk's, it reveals aspects of the problem that are not brought to light by the coloring argument.

Define a **fan** as a triangulation with one vertex (the fan **center**) shared by all triangles. Chvátal took as his induction hypothesis this statement:

> *Induction Hypothesis*: Every triangulation of an n-gon can be partitioned into $g \leq \lfloor n/3 \rfloor$ fans.

For the basis, note that $n \geq 3$ since we start with an n-gon, and that there is just a single triangulation possible when $n = 3, 4$, and 5, each of which is a fan. Thus the induction hypothesis holds for $n < 6$.

Given a triangulation with $n \geq 6$, our approach will be to remove part of the triangulation, apply the induction hypothesis, and then put back the deleted piece. We know that there is always a diagonal (in fact, there are always at least two) that partitions off a single triangle. But note that this only reduces n by 1, and if we were unlucky enough to start with $n \equiv 1$ or 2 (mod) 3, then the induction hypothesis partitions into $g = \lfloor (n-1)/3 \rfloor = \lfloor n/3 \rfloor$ fans, and we will in general end up with $g + 1$ fans when we put back the removed triangle. The moral is that, in order to make induction work with the formula $\lfloor n/3 \rfloor$, we have to reduce n by at least 3 so the induction hypothesis will yield less than g fans, allowing the grouping of the removed triangles into a fan.

So the question naturally arises: does there always exist a diagonal that partitions off 4 edges of the polygon, and therefore reduces n by 3? The answer is no! Chvátal's brilliant idea was to realize that there is always a diagonal that cuts off 4 or 5 or 6 edges:

Lemma 1 (Chvátal 1975) *For any triangulation of an n-gon with $n \geq 6$, there always exists a diagonal d that cuts off exactly 4, 5 or 6 edges.*

PROOF: Choose d to be a diagonal that separates off a minimum number of polygon edges that is at least 4. Let $k \geq 4$ be the minimum number, and label the vertices of the polygon $0, 1, \ldots, n-1$ such that d is $(0, k)$. d must support a triangle T whose apex is at some vertex t with $0 \leq t \leq k$. Since $(0, t)$ and (k, t) each cut off fewer than k edges, by the minimality of k we have $t \geq 3$ and $k - t \leq 3$. Adding these two inequalities yields $k \geq 6$.

Now the plan is to apply the induction hypothesis to the portion on the other side of the special diagonal d. Let G_1 be the triangulation partitioned off by d; it has $k+1$ boundary edges and hence is a $(k+1)$-gon. Let G_2 be the remainder of the original triangulation, sharing d; it has $n - k + 1$ vertices. The induction hypothesis says that G_2 may be partitioned into $g' = \lfloor (n-k+1)/3 \rfloor$ fans. Since $k \geq 4, g' \leq \lfloor (n-3)/3 \rfloor = \lfloor n/3 \rfloor - 1$. Thus, in order to establish the theorem, we have to show that G_1 need only add one more fan to the partition. We will consider each possible value of k in turn.

Case 1 ($k = 4$). G_1 is a 5-gon. We already observed that every pentagon is a fan. Therefore, G has been partitioned into $\lfloor n/3 \rfloor - 1 + 1 = \lfloor n/3 \rfloor$ fans.

Case 2 ($k = 5$). G_1 is an 6-gon. Consider the triangle T of G_1 supported by d, with its apex at t. We cannot have $t = 1$ or $t = 4$, as then the diagonals $(0, t)$ or $(5, t)$ [respectively] would cut off just 4 edges, violating the assumed minimality of $k = 3$. The cases $t = 2$ and $t = 3$ are clearly symmetrical, so assume without loss of generality that $t = 2$. Now the quadrilateral $(2, 3, 4, 5)$ can be triangulated in two ways:

Case 2a. The diagonal $(2, 4)$ is present. Then G_1 is a fan, and we are finished.

Case 2b. The diagonal $(3, 5)$ is present. Form the graph G_0 as the union of G_2 and T. G_0 has $n-5+1+1 = n-3$ edges. Apply the induction hypothesis to it, partitioning it into $g' = \lfloor (n-3)/3 \rfloor = \lfloor n/3 \rfloor - 1$ fans. Now T must be part of a fan F in the partition of G_0, and the center of F must be at one of T's vertices:

Case 2b.1. F is centered at 0 or 2. Then merge $(0, 1, 2)$ into F, and make $(2, 3, 4, 5)$ its own fan. Now all of G is covered with $g' + 1 = \lfloor n/3 \rfloor$ fans.

Case 2b.2. F is centered at 5. Merge both $(2, 3, 5)$ and $(3, 4, 5)$ into F, and make $(0, 1, 2)$ a separate fan. The result is $g' + 1$ fans.

Case 3 ($k = 6$). G_1 is a 7-gon. The tip t of the triangle T supported by d cannot be at 1, 2, 4, 5, as then a diagonal would exist that cuts off

$4 \leq k < 6$ edges, contradicting the minimality of k. Thus $t = 3$. Each of the two quadrilaterals $(0, 1, 2, 3)$ and $(3, 4, 5, 6)$ has two possible triangulations, leading to four subcases.

Case 3a. The diagonals $(3, 1)$ and $(3, 5)$ are present. Then G_1 is a fan centered at 3, and we are finished.

Case 3b. The diagonals $(0, 2)$ and $(3, 5)$ are present. Join the quadrilateral $(0, 2, 3, 6)$ to G_2 to form a polygon G_0 with $n - 6 + 1 + 2 = n - 3$ vertices, which by the induction hypothesis can be partitioned into $g' = \lfloor n/3 \rfloor - 1$ fans. Let F be the fan of this partition to which the triangle $(0, 2, 3)$ belongs. The center of F must be at one of its vertices:

Case 3b.1. F is centered at 0 or 2. Merge $(0, 1, 2)$ into F and make $(3, 4, 5, 6)$ a separate fan.

Case 3b.2. F is centered at 3. Merge $(3, 4, 5, 6)$ into F, and make $(0, 1, 2)$ a separate fan.

In all cases, G is partitioned into $g' + 1 = \lfloor n/3 \rfloor$ fans.

Case 3c. The diagonals $(1, 3)$ and $(4, 6)$ are present. This is the mirror image of Case 3b.

Case 3d. The diagonals $(0, 2)$ and $(4, 6)$ are present. Merge T with G_2 to form a polygon G_0 of $n - 6 + 1 + 1 = n - 4$ vertices. Applying the induction hypothesis partitions G_0 into $g' = \lfloor (n - 4)/3 \rfloor \leq \lfloor n/3 \rfloor - 1$ fans. Let F be the fan of the partition containing T.

Case 3d.1. F is centered at 0. Merge the quadrilateral $(0, 1, 2, 3)$ into F, and make $(3, 4, 5, 6)$ a separate fan.

Case 3d.2. F is centered at 3. Since all of G_2 is behind the $d = (0, 6)$ diagonal, it is clear that we can just as well consider F to be centered at 0, falling into Case 3d.1.

Case 3d.3. F is centered at 6. This is the mirror image of Case 3d.1.

In all cases, G is partitioned into $g' + 1 = \lfloor n/3 \rfloor$ fans. This completes the proof. □

Placing guards at the fan centeres establishes the theorem:

Theorem 1 (Chvátal's Art Gallery Theorem 1975) $\lfloor n/3 \rfloor$ *guards are occasionally necessary and always sufficient to see the entire interior of a polygon of* n *edges.*

Note that both Chvátal's and Fisk's proofs incidentally establish by construction that the guards can be chosen to be vertex guards. However it is an easy task to find an example where one point guard is sufficient but more than one vertex guard is needed to cover the polygon.

The material in this section comes from O'Rourke [42].

1.2 Sorting/Order Statistics

We shall in this section briefly discuss the problem of sorting a finite set that has an order relation. By sorting a sequence we mean rearranging the elements so that they appear in non-increasing or non-decreasing order.

A significant portion of commercial data processing involves sorting large quantities of data. Therefore a large number of sorting algorithms have been developed, see Knuth [32].

There are two classes of sorting algorithms. The first class assumes no structure on the elements to be sorted. The basic operation is a comparison between a pair of elements. The second class of algorithms makes use of the structure of the set to be sorted. We will as an example of this discuss the sorting of a set of integers in a fixed range.

A **partial order** on a set S is a relation R such that for each a, b and c in S:

1. aRa is true (R is reflexive);

2. aRb and bRc imply aRc (R is transitive);

3. aRb and bRa imply $a = b$ (R is antisymmetric).

A **linear order** or **total order** on a set S is a partial order such that for every pair of elements a, b we have that aRb or bRa. The relation $\leq$ on the integers is a total order but set inclusion is a partial order that in general is not a total order.

The sorting problem can be formulated as follows. We are given a sequence of n elements $a_1, a_2, \ldots, a_n$ from a set having a total order, which we shall denote by $\leq$. We want to find a permutation π of these n elements such that $a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}$ appear in a non-decreasing order, i.e.

$$a_{\pi(1)} \leq a_{\pi(2)} \leq \ldots \leq a_{\pi(n)}.$$

Knuth has introduced a notational device that distinguishes between upper and lower bounds, which we adopt.

- $O(f(n))$ denotes the set of all functions $g(n)$ such that there exist positive constants C and n_0 with

  $$|g(n)| \leq Cf(n)$$

  for all $n \geq n_0$.

- $\Omega(f(n))$ denotes the set of all functions $g(n)$ such that there exist positive constants C and n_0 with

  $$g(n) \geq Cf(n)$$

  for all $n \geq n_0$.

- $\theta(f(n))$ denotes the set of all functions $g(n)$ such that there exist positive constants C_1, C_2 and n_0 with

  $$C_1 f(n) \leq g(n) \leq C_2 f(n)$$

  for all $n \geq n_0$.

- $o(f(n))$ denotes the set of all functions $g(n)$ such that for all positive constants C there is an n_0 with

  $$|g(n)| \leq Cf(n)$$

  for all $n \geq n_0$.

We first show that any algorithm which sorts by comparison only, must on some sequence of length n use at least $\Omega(n \log n)$ comparisons.

The problem of sorting by comparisons can be expressed in other ways. Given a set of distinct weights and a balance scale, we can ask for the least number of weightings necessary to completely rank the weights in order of magnitude, when the pans of the balance scale can each accommodate only one weight.

All n–elements sorting methods which satisfy the above constraints can be represented in terms of a binary tree structure. Each node contains two indices $[i, j]$ denoting a comparison of a_i versus a_j. The left outgoing edge

corresponds to the outcome $a_i \leq a_j$ and the right edge represents the case $a_j \leq a_i$. We represent the termination of the algorithm by a box containing the permutation of $a_1, a_2, \ldots, a_n$ that leads to the ordering

$$a_{\pi(1)} \leq a_{\pi(2)} \leq \ldots \leq a_{\pi(n)}.$$

figure

Theorem 2 *There is a constant $C > 0$ such that for any algorithm that sorts n elements by comparison only, there is a sequence of n elements for which the algorithm requires at least $C\, n \log n$ comparisons*

PROOF: In the above model for an algorithm we can think of the termination points of the algorithm as leaves. We claim that after k steps of the algorithm we have at most 2^k leaves. This is obviously true for $k = 1$ and if $k > 1$ we see that the tree is a union of two disjoint subtrees, each of which has height $k - 1$. An induction argument therefore establishes the claim. Let N be the minimum number of comparisons that are required for any algorithm. Since the number of permutations of n elements is $n!$ we must have

$$2^N \geq n! \geq (\frac{n}{2})^{\frac{n}{2}},$$

which shows the result. □

We remark that using a better estimate of $n!$ like Stirling's formula, one can show that

$$N \geq \frac{n \log n + \frac{1}{2} \log n - n}{\log 2}.$$

Suppose S_1 and S_2 are two sequences that have been sorted in increasing order. We can easily sort the sequence one gets by concatenating S_1 and S_2 by successively selecting the smallest remaining element from S_1 and S_2. (In order to make this into a well-defined algorithm we break the ties in favour of S_1) This algorithm requires at most $m+n-1$ comparisons, where m, n are the number of elements in S_1, S_2. This remark is the basis for the following **divide-and-conquer**–method.

Theorem 3 *Let $T(n)$ be the smallest number of comparisons that are needed to sort a sequence of n elements in increasing order. Then*

$$T(n) = \theta(n \log n).$$

PROOF: If n is even it follows from the above remark that

$$T(n) \leq 2T(\frac{n}{2}) + n - 1.$$

If n is odd we have that

$$T(n) \leq T(\frac{n-1}{2}) + T(\frac{n+1}{2}) + n - 1.$$

An easy induction argument shows that $T(n) = O(n \log n)$. Using Theorem 2 concludes the proof. □

If a sequence to be sorted has some additional features it may be sorted in linear time. We will only discuss one example of this.

Let S be a set of integers $a_1, a_2, \ldots, a_n$ in the range $\{0, 1, \ldots, m-1\}$, $m \geq 1, n \geq 1$. We shall now describe the **bucket sort**–method. For each $k \in \{0, 1, \ldots, m-1\}$ initialize a que – each such que is called a bucket. Scan the sequence from left to right and place the element a_i in the a_i–bucket. Concatenate the buckets by appending the content of $k+1$ to the end of que k. This yields the sorted sequence in $O(n+m)$ time. If $m = O(n)$ we can therefore sort the sequence in linear time $O(n)$.

We now turn to **order statistics**.

Let $S = \{a_j\}_{j=1}^n$ be a sequence of elements with a total ordering $\leq$. We will say that a_j is the kth **smallest element** if a_j is the smallest element in S with the property that there is a subset $J \subset \{1, 2, \ldots, n\}$ with $k-1$ elements such that

$$a_i \leq a_j \quad \textit{for all} \quad j \in J.$$

Notice that the index of the kth smallest element does not need to be unique.

For real numbers x let $\lceil x \rceil$ be the smallest integer ν such that $\nu \geq x$. ($\lfloor x \rfloor$ denotes the largest integer ν such that $\nu \leq x$) We recall that the choice $k = \lceil \frac{n}{2} \rceil$ corresponds to the **median** of S. By careful application of the divide-and-conquer strategy, often referred to as **prune and search**, we can find the kth smallest element of S in $O(n)$ time.

Theorem 4 *Let S be a sequence of n elements with a linear ordering $\leq$. Then there is an algorithm that finds the kth smallest element in S in linear time $O(n)$.*

PROOF: Let $\varphi(k, S)$ denote the kth smallest element in S. Let $\tau(k, n)$ be the shortest time for finding the kth smallest element and set

$$T(n) = \max\{\tau(k, n) : 1 \leq k \leq n\}.$$

Assume $n \geq 100$. Set $m = \lceil \frac{n}{5} \rceil$ and split S into sets with 5 elements each and possibly one with fewer elements than 5. Call these sets $E_1, \ldots, E_m$ with E_ν having 5 elements for $\nu = 1, 2, \ldots, m-1$. Let x_k be the median of E_k and set $X = \{x_k\}_{k=1}^m$. Let $q = \varphi(\lceil \frac{m}{2} \rceil, X)$ be the median of X and set $J_1 = \{l \in I_n : a_l \leq q \ \ and \ \ a_l \neq q\}$, $J_2 = \{l \in I_n : a_l = q\}$ and $J_3 = \{l \in I_n : q \leq a_l \ \ and \ \ a_l \neq q\}$. Here $I_n = \{1, 2, \ldots, n\}$. Let ν_i denote the number of elements of J_i, $i = 1, 2, 3$. We claim that ν_1 and ν_3 are each at most $\frac{3n}{4}$. Let $Q = \{j : 1 \leq j < m \ \ and \ \ q \leq x_j\}$. Notice that Q contains at least $\frac{n}{10}$ elements (almost). For each $j \in Q$ there are at least two additional elements in E_j that are $\geq q$. Hence the size of J_1 is at most $n - 3\frac{n}{10} \leq \frac{3n}{4}$. Similar arguments applies to J_3. We can now observe the following. Let $S_i = \{a_l : l \in J_i\}$, $i = 1, 2, 3$. If $\nu_1 \geq k$ then $\varphi(k, S) = \varphi(k, S_1)$. If $\nu_1 < k$ but $\nu_1 + \nu_2 \geq k$ then $\varphi(k, S) = q$. If $\nu_1 + \nu_2 < k$ then $\varphi(k, S) = \varphi(k - \nu_1 - \nu_2, S_3)$. Hence there is a constant $C > 0$ such that

$$T(n) \leq T(\lceil \frac{3n}{4} \rceil) + T(\lceil \frac{n}{5} \rceil) + Cn.$$

An easy induction argument yields that $T(n) = O(n)$. □

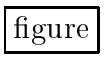
We remark that in most UNIX–systems one can find the quick-sort algorithm.

2 Triangulations

We will in this chapter be concerned with planar triangulations.

2.1 General Properties

In two dimensions, a **triangulation** $\mathcal{T}$ is a finite collection of closed triangles $\{T_i\}$ such that each T_i has a non-empty interior and the T_i:s have pairwise disjoint interiors. Moreover we also require that if two distinct triangles have more than one point in common the must share a common edge.

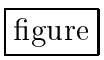


We will say that $\mathcal{T}$ is a triangulation of the set E if

$$E = \bigcup_{T \in \mathcal{T}} T.$$

Important problems are the existence of triangulations with prescribed properties and efficient algorithms for their construction. A general reference for this section is [45].

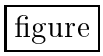
Theorem 5 *Let Ω be a closed simply connected polygonal domain and let V_Ω denote its set of vertices. Assume $W \subset \mathrm{Int}(\Omega)$ is a finite set. Then there exists a triangulation $\mathcal{T}$ of Ω whose set of vertices equals $V_\Omega \bigcup W$.*



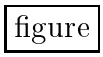
A **chord** or a **diagonal** of a polygon is a line segment between two vertices that lies inside the polygon and does not intersect the polygon's boundary except at the end points.

Lemma 2 *Every polygon with more than three sides has a chord.*

PROOF: (Theorem 5) We first see that there is a triangulation of Ω whose vertex set equals V_Ω. For once we have found a chord we can split the polygon in two and recursively triangulate each part. Next we add one point at a time. If P lies inside a triangle we split that triangle in three.



If P falls on a common edge of two triangles we split each of them in two triangles.



□

Every triangulation $\mathcal{T}$ gives rise to a graph on the vertex set of $\mathcal{T}$ by identifying edges of $\mathcal{T}$ with the edges of the graph. This graph is called the **skeleton** or **triangulation graph** of the triangulation.

Finally we remark that the number of edges in a triangulation of a polygonal domain Ω is uniquely determined by the number of the vertices at the boundary and the number of the inner vertices. More precisely, we have the following result which easily follows by applying Euler's formula.

Theorem 6 *Let Ω be a closed simply connected polygonal domain and let V_Ω denote its set of vertices. Assume $W \subset \mathrm{Int}(\Omega)$ is a finite set. Let $\mathcal{T}$ be any triangulation of Ω whose set of vertices equals $V_\Omega \bigcup W$. Then*[1]

[1]For a finite set U we let $|U|$ denote the number of elements in U.

1. *the number of triangles in $\mathcal{T}$ is equal to*

$$|V_\Omega| + 2|W| - 2.$$

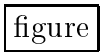
2. *the number of edges in $\mathcal{T}$, diagonals and boundary edges, is equal to*

$$2|V_\Omega| + 3|W| - 3.$$

How fast can we triangulate a polygon with N vertices? Chazelle [9] has established a linear time algorithm for this.

2.2 Transformations of Triangulations

Let $\mathcal{T}$ be a triangulation and assume that assume that $T_1, T_2 \in \mathcal{T}$ share a common edge and that $R = T_1 \bigcup T_2$ is a convex quadrilateral. By replacing the diagonal of R with the opposite diagonal we get a new triangulation with the same set of vertices.



This transformation is called **swapping of diagonals or edges**.

Suppose now that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two triangulations of a polygon with the same vertex sets. It is natural to ask if it is possible to transform $\mathcal{T}_1$ into $\mathcal{T}_2$ by only using diagonal swaps of convex quadrilaterals. The following result by Dyn/Goren/Rippa [18] answers this question.

Theorem 7 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangulations of a polygonal domain Ω. Assume $\mathcal{T}_1$ and $\mathcal{T}_2$ have the same set of vertices. Then it is possible to transform $\mathcal{T}_1$ into $\mathcal{T}_2$ by only swapping diagonals of convex quadrilaterals.*

We begin with the following easy lemma.

Lemma 3 *Let $\gamma = \{P_0, \ldots, P_{N-1}\}$ be a simple closed polygon. Then γ has at least three interior angles strictly less that π.*

PROOF: Let $\alpha_0, \ldots, \alpha_{N-1}$ denote the interior angles of γ. Then

$$\Sigma_{i=0}^{N-1}\alpha_i = (N-2)\pi.$$

Let M be the number of α_i:s that are strictly less than π. Then

$$(N-2)\pi \geq (N-M)\pi + \Sigma_{\alpha_i<\pi}\alpha_i > (N-M)\pi,$$

so $M > 2$. □

Lemma 4 *Let $\mathcal{T}$ be a triangulation of the closed set Ω. Denote by V the set of vertices of $\mathcal{T}$ and let $A, B \in V$ be two distinct vertices. Assume $(A,B) \subset \mathrm{Int}(\Omega)$ and $V \bigcap (A,B) = \emptyset$. Then there exists a triangulation $\mathcal{T}^*$ of Ω whose vertex set is V and such that $[A,B]$ is an edge of $\mathcal{T}^*$. The transformation from $\mathcal{T}$ to $\mathcal{T}^*$ can be achieved by swapping diagonals of convex quadrilaterals. Moreover only edges that intersect (A,B) are swapped during the transformation.*

PROOF: Assume that $[A,B]$ is not an edge[2] of $\mathcal{T}$. Set $\mathcal{F}_\mathcal{T} = \{T \in \mathcal{T} : \mathrm{Int}(T) \bigcap (A,B) \neq \emptyset\}$ and $W = \bigcup_{T\in\mathcal{F}_\mathcal{T}} T$. Then $[A,B] \subset W$. Since every point $P \in W$ can be connected to $[A,B]$ inside the triangle $T \in \mathcal{F}_\mathcal{T}$ that contains P we see that W is connected.

Let L be the line through A and B. Clearly $L \bigcap \partial W = \{A,B\}$ and $(L \setminus [A,B]) \bigcap W = \emptyset$. L separates the plane into two open half planes H_+ and H_-.

We now claim that W is simply connected. Let $\mathcal{F}_\mathcal{T} = \{F_i\}$ and notice that the closed intervals $J_k = T_k \bigcap [A,B]$ have pairwise disjoint interiors. We may assume that the ordering of the T_k:s has been chosen so that $\mathrm{dist}(A, J_k)$ increases with k. Set $W_n = \bigcup_{k=1}^n T_k$. The above argument shows that W_n is connected for every n. We also observe that $\partial T_{n+1} \setminus W_n$ is connected. Hence W_{n+1} is simply connected if W_n is simply connected. Therefore W is simply connected by induction.

Set $S = \partial W \bigcap V$. From the above follows that $W_+ = W \bigcap H_+$ is a polygon with vertices $(S \bigcap H_+) \bigcup \{A,B\}$.

The idea of the proof is now to construct a new triangulation $\mathcal{T}'$ such that $\mathcal{F}_{\mathcal{T}'}$ has fewer triangles than $\mathcal{F}_\mathcal{T}$.

Using Lemma 3 on the polygon W_+ we see that there is a $p \in S \bigcap H_+$ such that the interior angle to W at p is strictly less than π. Let $\mathcal{F}_\mathcal{T}(p)$ be

[2]We have adopted the convention that $[A,B]$ denotes the closed segment AB, i.e. with the endpoints A, B included, while (A,B) denotes the open segment $[A,B] \setminus \{A,B\}$.

the set of triangles $T \in \mathcal{F}_{\mathcal{T}}$ that have a vertex at p. Let $\mathcal{E}_{\mathcal{T}}(p)$ be the class of edges e in $\mathcal{T}$ such that $e \bigcap (A, B) \neq \emptyset$ and have end point at p. We have that $\mathcal{E}_{\mathcal{T}}(p) \neq \emptyset$ and if $e \in \mathcal{E}_{\mathcal{T}}(p)$ then e is the common edge of two triangles in $\mathcal{F}_{\mathcal{T}}(p)$.

Set $\mathcal{E}_{\mathcal{T}}(p) = \{e_1, \ldots, e_n\}$ and let q_i be the end point of e_i that is different from p. Notice that it may happen that $q_i \in \mathrm{Int}(W)$. We now give ∂W the counterclockwise ordering. Let p_0, p_1 be the elements in S that are immediately before and after p. We now assume that $q_1, \ldots, q_n$ are listed in counterclockwise order as end points of rays emanating from p. Since each q_i belongs to H_- and is contained inside the sector determined by p_0, p and p_1 it follows that $\{q_1, \ldots, q_n, p_0, p_1\}$ is a simple polygon Q.

Using Lemma 3 on Q we see that there is a k, $1 \leq k \leq n$, such that the interior angle of Q at q_k is strictly less than π. Let T_1 and T_2 be the triangles in $\mathcal{F}_{\mathcal{T}}(p)$ with the common edge e_k and set $R = T_1 \bigcup T_2$. Then the interior angle of R at p is strictly less than π. If $1 < k < n$ then R is contained in the sector determined by q_{k-1}, q_k and q_{k+1}. Hence the interior angle of R at q_k is strictly less than π in this case. If k equals 1 then one of the triangles T_1, T_2 must equal the triangle pp_1q_1. Hence in this case R is contained in the sector determined by p_1, q_1 and q_2. If $k = n$ then R is contained in the sector determined by q_{n-1}, q_n and p_0. Therefore the interior angle of R at q_k is strictly less than π. Hence R is a convex quadrilateral. Swapping diagonals in R gives us a new triangulation $\mathcal{T}^*$ such that $\mathcal{E}_{\mathcal{T}^*}(p)$ has one edge less than $\mathcal{E}_{\mathcal{T}}(p)$. Repeating this construction leads to a triangulation $\mathcal{T}'$ with $\mathcal{E}_{\mathcal{T}'}(p) = \emptyset$. Hence $\mathcal{F}_{\mathcal{T}'}$ has fewer triangles than $\mathcal{F}_{\mathcal{T}}$.

Iterating the above procedure leads to a triangulation $\mathcal{T}_1$ with $\mathcal{F}_{\mathcal{T}_1} = \emptyset$, i.e. $[A, B]$ is an edge in $\mathcal{T}_1$. The lemma is proved. □

PROOF: (Theorem 7) Let $e_1, \ldots, e_n$ be the edges in $\mathcal{T}_2$ that do not belong to the boundary of Ω. Let $\mathcal{T}^{(1)}$ be the triangulation one gets by using Lemma 4 to transform $\mathcal{T}_1$ into a triangulation that contains e_1. For $2 \leq k \leq n$ we let $\mathcal{T}^{(k)}$ be the triangulation one gets by using Lemma 4 to transform $\mathcal{T}^{(k-1)}$ into a triangulation that contains e_k. Since e_j and e_k do not intersect if $1 \leq j < k$ it follows from Lemma 4 that $e_1, \ldots, e_{k-1}$ belong to $\mathcal{T}^{(k)}$, $2 \leq k \leq n$. Hence $\mathcal{T}^{(n)}$ equals $\mathcal{T}_2$ which yields the theorem. □

3 Convex Hulls

We complete O'Rourke's presentation of convex hull algorithms, see [41], with the algorithm of Kirkpatrik-Seidel which connects several concepts treated above.

3.1 Kirkpatrik-Seidel's Ultimate Convex Hull Algorithm

This algorithm was published in 1986 in [30].

The basic idea is to develop a divide and conquer algorithm that runs in $O(n \log h)$ time, where h is the size of the hull e.g. the number of extreme edges. For simplicity, we will only concentrate on finding the upper chain of the convex hull, which connects the leftmost point with the rightmost point.

We will divide the points in P into two sets P_L and P_R of approximately equal size by using a vertical separating line. A few key observations are in order.

1. We can compute the upper supporting line, the **bridge**, *before* we compute the convex hulls of P_L and P_R, since these are functions of the point set, and not of the convex hulls of P_L and P_R. (Of course, the computation will be a little trickier since we cannot use the convex hull to find the bridge, as we did in the regular divide and conquer algorithm.)

2. We can delete all points that are immediately "below" the bridge. Specifically, if the bridge joins points p and q, we can delete all points with x coordinates between p and q. We recurse on the remaining points to find the upper chain of P'_L and P'_R (remaining sets of points). This will potentially reduce the number of points we recurse on.

There are a few things we need to do to implement the above ideas. The first is that we cannot afford to presort the points before starting the algorithm, so that for each divide step we have to invoke a linear time median finding algorithm to split S into equal pieces.

The second challenge is to find the bridge in $O(n)$ time. This will be done by a prune and search argument. In this method, we spend cn time, and we either find what we are looking for, or (for example) delete half the points from consideration, and recurse. The running time for such a scheme is

$$cn + c\frac{n}{2} + c\frac{n}{4} \ldots c = O(n).$$

Observe that this actually works, so long as we eliminate a constant fraction of the points!

Before we get into details about how the bridge is actually computed, we will try to analyse the running time as a function of h, the size of the upper chain of the hull. If $h = 1$ in any recursive call, we compute a bridge and realize that the bridge connects the leftmost and rightmost points, and all other points can be eliminated immediately. Hence this takes $O(n)$ time.

If the upper chain of P'_R has size h_R and the upper chain of P'_L has size h_L, then we have that $h = 1 + h_R + h_L$. The running time of the algorithm can be expressed as $T(n, h)$.

$$T(n, h) \leq cn \quad \text{if} \quad h = 1$$
$$T(n, h) \leq cn + T\Big(\frac{n}{2}, h_L\Big) + T\Big(\frac{n}{2}, h_R\Big).$$

We need to argue that regardless of the split of h between h_L and h_R, the algorithm runs in $O(n \log h)$ time. This is easy to prove by induction. Assume $T(m, h') \leq cm \log h'$ for $m < n$ and $1 < h' < h$. This implies

$$T(n, h) \leq cn + c\frac{n}{2} \log h_L + c\frac{n}{2} \log h_R.$$

Now use the fact that $\frac{1}{2}(\log h_L + \log h_R) \leq \log \frac{h_L + h_R}{2}$, i.e. that the log-function is a concave function. This gives

$$T(n, h) \leq cn + c\frac{n}{2}\Big(2 \log \Big(\frac{h_L + h_R}{2}\Big)\Big) \leq cn + cn(\log h - 1) \leq cn \log h.$$

It remains to indicate how to compute the **bridge** in $O(n)$ time.

Assume that the bridge has slope K^*. We will "guess" a value for the slope K, and try to figure out how K^* compares to K. Through each point p_i in P, let the line of slope K be ℓ_i. The line ℓ_j which has all the points of P "below" it is the line we are looking for – this line has the highest intercept with the dividing line (or the y axis, since they are all parallel). In $O(n)$ time we can compute which line has this property. (This line may not be unique.) If the line is defined by a point from P_L and a point from P_R this is the bridge we are looking for. If the line is defined *only* by points of P_L then $K^* < K$, and if the line is defined *only* by points of P_R then $K^* > K$.

Figure

In any case, this gives us some information about K^*. To apply the "prune and search" paradigm, we need to somehow discard a constant fraction of the points to continue looking for the bridge. We might "guess" the correct slope, and as soon as we do that the algorithm stops. If our guess is incorrect, we recurse on a constant fraction of the points. The running time is a decreasing geometric series and converges to $O(n)$.

We now outline the method for the pruning step. Consider a pair of points $a, b \in P$. Assume that a is to the left of b. Let the slope defined by this pair of points be K'. Suppose $K^* < K$ and $K \leq K'$, then $K^* < K'$. In this case we claim that a cannot be a bridge vertex. Suppose $K^* > K$ and $K \geq K'$, then $K^* > K'$. In this case we claim that b cannot be a bridge vertex. In either case, we are able to eliminate points as being candidate vertices.

How do we guarantee that we can find sufficiently many lines that have slope less than K, and sufficiently many lines with slope more than K? To achieve this, we arbitrarily pair up the points in set P, and compute their slopes. We then select the median slope of this set of $\frac{n}{2}$ lines, and use that as our guess for the slope of the bridge. We either guess the slope correctly and stop, or in either case we eliminate one point from $\frac{n}{4}$ pair's. We are left with at most $\frac{3n}{4}$ points and recurse on this set of points. This completes the prune and search algorithm.

Figure

4 Delaunay Triangulations/Voronoi Diagrams

Delaunay triangulations are named after the Russian mathematician Boris Delone. See [17] written in memory of George Voronoi.

4.1 Delaunay Triangulation: Existence and Uniqueness

Let $S \subset \mathbf{R}^2$ be a finite set of points that does not lie on a line. We say that $\mathcal{T}$ is a **Delaunay triangulation** if $\mathcal{T}$ is a triangulation of the convex hull of S with vertex set S and the circumcircles of the triangles in $\mathcal{T}$ newer contain a point from S in their interiors.

We shall now prove the existence of Delaunay triangulations. We will do this by taking the convex hull of the lift of S to the graph of $f(x,y) = x^2+y^2$. The crucial property of the graph of f that we will need is the following observation.

Proposition 1 *Let $f(x,y) = x^2 + y^2$ and $g(x,y) = \langle (x,y), v\rangle + a$, $v \in \mathbf{R}^2$ and $a \in \mathbf{R}$. Then $f(x,y) < g(x,y)$ if and only if $|(x,y) - \frac{v}{2}|^2 < L$, where $L = a + \frac{|v|^2}{4}$.*

Proof: Trivial. □

The significance of this result is that the set of points in $\mathbf{R}^3$ where the graph of f is below a plane always projects to the interior of a circle in $\mathbf{R}^2$.

Theorem 8 (Existence) *Let $S \subset \mathbf{R}^2$ be a finite set that does not lie on a line. Then S has a Delaunay triangulation.*

Proof: Set $f(x,y) = x^2+y^2$ and put $S_f = \{(x,y,f(x,y)) : (x,y) \in S\}$. Let now E be the convex hull of S_f in $\mathbf{R}^3$. The lower part of the boundary of E is the union of closed convex polygons F_i. For each F_i there is a $v_i \in \mathbf{R}^2$ and an $a_i \in \mathbf{R}$ such that if $g_i(x,y) = \langle (x,y), v_i\rangle + a_i$ then $f(x,y) \leq g_i(x,y)$ whenever $(x,y,z) \in F_i$ for some $z \in \mathbf{R}$. If $(x,y,z) \in F_i \bigcap S_f$ then $f(x,y) = g_i(x,y)$. Let G_i be the projection of F_i onto $\mathbf{R}^2$. From Proposition 1 follows that $G_i \bigcap S$ is contained in a circle C_i. Also C_i contains no point from S in its interior. Triangulating each G_i now yields the existence of a Delaunay triangulation. □

We shall next show how the edge-swapping transformation described in the previous section can be adapted to construct a Delaunay triangulation. The algorithm starts with any triangulation $\mathcal{T}$ of the convex hull of S. For a triangle T we let $C(T)$ and $\omega(T)$ denote the circumcircle of T and the closure of its interior. If e is an interior edge of $\mathcal{T}$ we let $R(e)$ denote the union of the triangles in $\mathcal{T}$ with the common edge e. If there is an interior edge e such that $R(e)$ is strictly convex and some vertex of $R(e)$ is contained in the interior of $C(T)$ for some triangle $T \in \mathcal{T}$ with edge e as an edge then we swap the edge. We call this the **flip algorithm**. We refer to [4] for the following result.

Theorem 9 *The flip algorithm terminates at a Delaunay triangulation.*

For the proof of the theorem we need the following results.

Lemma 5 *Let $T_1 = ABC$ and $T_2 = BCD$ be two triangles with disjoint interiors and a common edge BC. Assume that the quadrilateral $R = T_1 \bigcup T_2$ is not strictly convex. Then D lies outside $C(T_1)$ and A lies outside $C(T_2)$.*

figure

PROOF: From the assumption follows that R has at least one interior angle $\geq \pi$. Since the interior angles of R at A and D are strictly less that π we may assume that the interior angle at B is $\geq \pi$. We will argue by contradiction. Assume that D fall inside $C(T_1)$. Let D_1 be the intersection between $C(T_1)$ and the ray emanating from B that goes through D. Then the polygon ABD_1C has all its vertices on $C(T_1)$ and is therefore strictly convex. This contradicts the fact that $ABDC$ and ABD_1C have the same interior angle at B. By symmetry we are done. □

Proposition 2 *Suppose T_1, T_2 are two triangles with disjoint interiors and a common edge e. Let A_1, A_2 be the vertices of T_1, T_2 that are not on e. Assume that A_1 falls outside $C(T_2)$ and that A_2 falls outside $C(T_1)$. Let L*

be the line through e. Assume that L separates the point P from T_1 and that P falls inside $C(T_1)$. Then P falls inside $C(T_2)$.

PROOF: We need only to treat the case when P does not belong to L. Let H be the half plane determined by L that contains P. By elementary geometry we have that $\omega(T_1) \bigcap H \subset \omega(T_2) \bigcap H$. □

PROOF: (Proof of Theorem 9) Let $\mathcal{E}(\mathcal{T})$ be the set of interior edges e such that if P is any vertex of $R(e)$ and if $T \in \mathcal{T}$ is any triangle with e as an edge then P belongs to the closure of the exterior of $\rho(P)$. From Corollary 3 follows that $\mathcal{E}(\mathcal{T})$ increases by one element with every step of the flip algorithm. Hence the flip algorithm terminates.

We now let $\mathcal{T}$ be a triangulation that is left unchanged by the flip algorithm. Let $T_1, T_2 \in \mathcal{T}$ be two triangles with a common edge e. Let A_1, A_2 be the vertices of T_1, T_2 that do not lie on e. Since the flip algorithm has terminated it follows from Corollary 3 or Lemma 5 that A_1 is outside $C(T_2)$ and A_2 is outside $C(T_1)$.

Assume that $\mathcal{T}$ is not a Delaunay triangulation. Then there exists a $T \in \mathcal{T}$ and a vertex P for $\mathcal{T}$ such that P is in the interior of $C(T)$. Clearly $P \notin T$. Then there is an edge e of T such that if L denotes the line through e then L separates T from P. We notice that P can not lie on L. Let A be one of the end points of e. We may assume that (A, P) does not contain any vertex of $\mathcal{T}$ for otherwise we replace P by the vertex of $\mathcal{T}$ that is closest to A.

Let $T^* \in \mathcal{T}$ be the triangle different from T that also has e as an edge. Then T^* and P lie on the same side of L. From Proposition 2 follows that P is contained in the interior of $C(T^*)$.

We begin by assuming that $\text{Int}(T^*) \bigcap (A, P) \neq \emptyset$. In this case $[A, P]$ is not an edge for $\mathcal{T}$. Let $T_1, \ldots, T_n$ be the triangles in $\mathcal{T}$ that intersect (A, P). We let them be ordered such that $\text{dist}(T_k \bigcap (A, P), A)$ increases with k. Then $T_1 = T^*$. Let L_k be the line through the common edge of T_k and T_{k+1}. We observe that L_k separates P from T_k. By repeated application of Proposition 2 we see that P is in the interior of $C(T_k)$ for all k, $1 \leq k \leq n$. This contradicts the fact that $P \in T_n$.

Let W be the closed convex sector with apex A determined by e and $[A, P]$. Let $T_1, \ldots, T_n$ be the triangles in $\mathcal{T}$ with apex at A that are contained in W. We determine the ordering by requiring that $T_1 = T^*$ with T_k and T_{k+1} having a common edge e_k. Let L_k be the line determined by e_k. Then

L_k separates P from T_k so P must be contained in the interior of $C(T_k)$ for all k, $1 \leq k \leq n$. Let e_n be the edge of T_n with an end in A that is different from e_{n-1}. Let $T_{n+1} \in \mathcal{T}$ be the triangle different from T_n that has e_n as an edge. Since P belongs to the interior of $C(T_n)$ it follows that $\mathrm{Int}(T_{n+1}) \bigcap (A, P) \neq \emptyset$. This is impossible by our previous argument. This completes the proof of the theorem. □

Theorem 10 *Let $S \subset \mathbf{R}^2$ be a finite set that does not lie on a line. Suppose $a, b \in S$, $a \neq b$, and that there is a circle C such that $a, b \in S$ and there is no point from S in the interior of C. Then there is a Delaunay triangulation $\mathcal{T}$ of S such that $[a, b]$ is an edge in $\mathcal{T}$. Assume that $a, b, c \in S$ are the vertices of the triangle T. If $C(T)$ contains no points from S in its interior then T is a triangle in some Delaunay triangulation of S that can be constructed as in Theorem 8.*

PROOF: We begin by establishing the second part of the theorem. Let P_a, P_b and P_c be the lifts of a, b and c to the graph of the function $f(x, y) = x^2 + y^2$. Then there is a unique plane H passing through P_a, P_b and P_c. From Proposition 1 follows that the lift S_f of S to the graph of f lies on one side of H. Hence T belongs to some Delaunay triangulation of S.

We can now prove the first part of the theorem by observing that the circle C can be modified so as to contain at least three points from S without having any point of S in its interior. □

Theorem 11 (Uniqueness) *Let $S \subset \mathbf{R}^2$ be a finite set that does not lie on a line. Assume that no four points in S belong to a circle.Then S has a unique Delaunay triangulation.*

PROOF: Set $f(x, y) = x^2 + y^2$ and let S_f be the lift of S to the graph of f. The lower part of the boundary of the convex hull of S_f is the union of convex polygons F_i. From Proposition 1 follows that each F_i has exactly three vertices. Hence there is a unique Delaunay triangulation $\mathcal{T}^*$ constructed by Theorem 8. If $\mathcal{T}$ is any Delaunay triangulation of S then every triangle from $\mathcal{T}$ belongs to $\mathcal{T}^*$ by the previous theorem. □

We remark that by a similar argument one can show that any Delaunay triangulation is of the form given by Theorem 8. We leave the verification of this to the reader.

4.2 Delaunay Triangulation from a Stereographic Transformation

We shall in this section indicate how to construct Delaunay triangulations by stereographic transformations onto a sphere. A reference for this result is [7].

Let

$$M(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^2}, \quad \mathbf{x} = (x, y, z) \in \mathbf{R}^3 \setminus \{0\}.$$

It is easily seen that M maps the plane $H = \{(x, y, -1) : (x, y) \in \mathbf{R}^2\}$ onto $\Omega \setminus \{0\}$, where Ω is the sphere

$$\Omega = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x} - (0, 0, -\frac{1}{2})| = \frac{1}{2}\}.$$

We also remark that M maps circles and lines in H onto circles in Ω and conversely.

We shall now construct a triangulation using the following procedure.

Let $S \subset H$ be a finite set. Let E_S denote the convex hull of the set $M(S) \subset \Omega \setminus \{0\}$. Then E_S is a convex polyhedron. We now let E_S^+ denote the class of faces $\{F_i\}$ of E_S such that 0 and E_S lie on the same side of the plane containing F_i.

We denote by V_i the vertex set of F_i, i.e. $V_i = F_i \bigcap M(S)$. Hence $M(V_i)$ is contained in a circle. Choose now a triangulation of the convex hull of $M(V_i)$. We let $\mathcal{T}_M$ be the collection of triangles one gets by letting F_i vary over all faces in E_S^+ and making one choice of the triangulation of the convex hull of $M(V_i)$. We call $\mathcal{T}_M$ a **triangulation** obtained by **stereographic projection**.

Theorem 12 *Let $S \subset \{(x, y, -1) : (x, y) \in \mathbf{R}^2\}$ be a finite set that does not lie on a line. Then every triangulation one gets by stereographic projection is a Delaunay triangulation.*

PROOF: Define

$$\tilde{M}(\mathbf{x}) = -\frac{\mathbf{x}}{z}, \quad \mathbf{x} = (x, y, z) \in \tilde{\Omega} \equiv \{(x, y, z) \in \mathbf{R}^3 : -1 \leq z < 0\}.$$

Notice that $M(\mathbf{x}) = \tilde{M}(\mathbf{x})$ for all $\mathbf{x} \in \Omega \setminus \{0\}$. Since $\tilde{M}$ is one–to–one on the lower part E_S^+ of E_S and $\tilde{M}(F_i)$ equals the convex hull of $M(V_i)$ we see that

$\mathcal{T}_M$ is a triangulation of the convex hull of S. From the mapping properties of M it follows that for all $T \in \mathcal{T}_M$ the intersection between S and the interior of the circumcircle of T is empty. $\square$

4.3 Delaunay Triangulations minimize the Dirichlet Integral

We shall in this section study the Dirichlet integral of piecewise linear functions. Let $\mathcal{T}$ be a triangulation with vertex set S. Assume $f : S \to \mathbf{R}$ and let $f_{\mathcal{T}}$ denote the piecewise linear extension of f to $\bigcup_{T\in\mathcal{T}}$ that is linear on each $T \in \mathcal{T}$. We let

$$\mathcal{D}(f, \mathcal{T}) = \Sigma_{T\in\mathcal{T}} \int\int_T |\nabla f_{\mathcal{T}}|^2 \, dxdy$$

denote the **Dirichlet integral of** f.

We shall now establish the following remarkable extremal property of Delaunay triangulation due to Rippa [47].

Theorem 13 *Let $\mathcal{T}$ be a Delaunay triangulation of the set V. Let $f : V \to \mathbf{R}$ be any function. Assume that $\mathcal{T}^*$ is a triangulation of the convex hull of V with vertex set V. Then*

$$\mathcal{D}(f, \mathcal{T}) \leq \mathcal{D}(f, \mathcal{T}^*).$$

If $\mathcal{D}(f, \mathcal{T}) = \mathcal{D}(f, \mathcal{T}^*)$ then Lemma 13 can be used to analyze the structure of f and the relationship between $\mathcal{T}$ and $\mathcal{T}^*$. For an example of this see the proof of Theorem 17.

To give the proof of the theorem we need some lemmas. We give the following classical result for the convenience of the reader.

Lemma 6 (Ptolemy's First Theorem) *Suppose $ABCD$ is a quadrilateral inscribed in a circle. Then*

$$|[A, C]| \cdot |[B, D]| = |[A, B]| \cdot |[C, D]| + |[A, D]| \cdot |[B, C]|.$$

PROOF: We will use the following identity by Euler. Let $a, b, c, d \in \mathbf{C}$. Then

$$(d - a)(b - c) + (d - b)(c - a) + (d - c)(a - b) = 0.$$

We remark that if a, b, c, d lie on a line and are sequentially ordered then

$$|a-c| \cdot |b-d| = |a-d| \cdot |b-c| + |a-b| \cdot |c-d|.$$

We may assume that the circumcircle of the quadrilateral goes through the origin and that the origin is outside the quadrilateral. Setting

$$a = \frac{1}{A},\ b = \frac{1}{B},\ c = \frac{1}{C},\ d = \frac{1}{D}$$

yields the lemma. □

Lemma 7 *Suppose $ABCD$ is a strictly convex quadrilateral. Then*

$$|[A,C]| \cdot |[B,D]| \leq |[A,B]| \cdot |[C,D]| + |[A,D]| \cdot |[B,C]|$$

with equality if and only if the vertices fall on a circle.

Proof: Assume that C lies outside the closed disk Δ, whose boundary contains A, B and D. Let C' be the intersection between $\partial\Delta$ and $[A,C]$.

figure

Select a coordinate system such that C' falls at the origin and C falls on the positive x-axis. For $x > 0$, let $P = (x, 0)$ and set

$$f(x) = |[A,B]| \cdot |[C,D]| + |[A,D]| \cdot |[B,C]| - |[A,C]| \cdot |[B,D]|.$$

Let θ and θ' be the angles APD and $AC'D$. Also let ϕ and ϕ' be the angles APB and $AC'B$. Then

$$0 < \theta < \theta' < \frac{\pi}{2}, \quad 0 < \phi < \phi' < \frac{\pi}{2}.$$

Now if $x > 0$ then

$$\begin{aligned} \frac{df}{dx} &= |[A,B]| \cos\theta + |[A,D]| \cos\phi - |[B,D]| \\ &> |[A,B]| \cos\theta' + |[A,D]| \cos\phi' - |[B,D]| = 0 \end{aligned}$$

since the angles ABD and ADB equal θ' and ϕ' respectively. Since $f(0) = 0$ the lemma follows. □

Lemma 8 *Let T be a triangle with vertices A, B, C and denote by A_m, C_m the midpoints of $[A, B]$ and $[B, C]$. Let ω_A and ω_C be the closed disks centered at A_m and C_m and passing through B. Denote by S the closed sector with apex B determined by T. Then*

$$\omega_A \bigcap \omega_C \bigcap S \subset T.$$

PROOF: We note that $\partial\omega_A \bigcap \partial\omega_C$ consists of exactly two points B and Q. Let Q_m be the midpoint of $[B, Q]$. Let L_m be the line through A_m and C_m.

figure

Then L_m intersects $[B, Q]$ perpendicularly at Q_m. If L denotes the line through Q that is parallel to L_m then A and C lie on L. Let H be the closed half plane determined by L that contains B. Then

$$\omega_A \bigcap \omega_C \bigcap S \subset H.$$

Hence

$$\omega_A \bigcap \omega_C \bigcap S \subset H \bigcap S = T.$$

□

If Q is the vertex of a polygon Γ we let $\omega(Q)$ denote the closed disk whose boundary goes through Q and its immediate neighbors. Also, $R(Q)$ will denote the radius of $\omega(Q)$.

Lemma 9 *Let $ABCD$ be a strictly convex quadrilateral. Assume that $C \notin \omega(A)$. Then*

$$\min(R(A), R(C)) \leq \min(R(B), R(D)). \tag{1}$$

PROOF: For a vertex P we let z_P denote the center of $\omega(P)$. For two points P and Q we let $\Delta_{P,Q}$ be the closed disk that has $[P,Q]$ as a diameter.

From the assumptions follows that $A \notin \omega(C)$ but $\omega(B)$ and $\omega(D)$ contain all vertices.

Assume that (1) is false, say

$$R(B) < \min(R(A), R(C)).$$

Let L be the line through $[A,B]$ and let $H = H_{A,B}$ be the closed half plane determined by L that contains C. Let H_- be the complement of H. We now claim that

$$H_{A,B} \bigcap \omega(A) \bigcap \omega(B) \subset \Delta_{A,B}. \tag{2}$$

If z_A or z_B belong to H_- then (2) is obvious. If z_A and z_B both were in $H_{A,B}$ then clearly $H_{A,B} \bigcap \omega(B) \subset H_{A,B} \bigcap \omega(A)$. However $C \in H_{A,B} \bigcap \omega(B)$ but $C \notin H_{A,B} \bigcap \omega(A)$ so this is impossible. Hence (2) follows.

Let $H_{B,C}$ be the closed half plane that contains A and has the line through $[B,C]$ as its boundary. By the same argument we have

$$H_{B,C} \bigcap \omega(B) \bigcap \omega(C) \subset \Delta_{B,C}. \tag{3}$$

Hence $D \in \Delta_{A,B} \bigcap \Delta_{B,C}$. From the Lemma 8 follows that D belongs to the triangle ABC, which violates the assumption that $ABCD$ is strictly convex. By this contradiction (3) follows. □

Lemma 10 *Let $ABCD$ be a strictly convex quadrilateral. Assume that $C \notin \omega(A)$. Then*

$$\max(R(A), R(C)) \leq \max(R(B), R(D)). \tag{4}$$

PROOF: Notice that $A \notin \omega(C)$. We shall argue by contradiction. Assume that the conclusion fails with, say,

$$R(A) > \max(R(B), R(D)).$$

For a vertex P let z_P denote the center of $\omega(P)$. Let L be the line through $[A,B]$ and denote by H_B the closed half plane determined by L that contains C. We claim that $z_A \notin H_B$. For if $z_A \in H_B$ then $H_B \bigcap \omega(B) \subset H_B \bigcap \omega(A)$ by the assumption that $R(A) > R(B)$. This is impossible since $C \notin \omega(A)$. Hence $z_A \in H_B^-$, where H_B^- is the complement of H_B.

By the same argument it follows that $z_A \in H_D^-$, where H_D^- is the plane determined by the line through $[A,D]$ that does not contain C. Hence $z_A \in H_B^- \bigcap H_D^-$, which is impossible. □

Lemma 11 *Let f be linear in the triangle ABC. Assume $f(A) = 1$, $f(B) = f(C) = 0$. Then*

$$\int\int_{ABC} |\nabla f|^2 \, dxdy = \frac{|[B,C]|^2}{4|ABC|},$$

where $|ABC|$ is the area of the triangle.

PROOF: Letting h be the height of the triangle from the vertex A one has that $|\nabla f| = \frac{1}{h}$. Hence

$$\int\int_{ABC} |\nabla f|^2 \, dxdy = \frac{|ABC|}{h^2} = \frac{|[B,C]|}{2h} = \frac{|[B,C]|^2}{4|ABC|}.$$

□

Lemma 12 *Let $ABCD$ be a strictly convex quadrilateral. Let $\mathcal{T}_1$ be the triangulation of $ABCD$ with triangles ABC and ADC. Let $\mathcal{T}_2$ be the triangulation of $ABCD$ with triangles ABD and BDC. Let $S = \{A, B, C, D\}$ and suppose $f : S \to \mathbf{R}$ satisfies $f(B) = f(C) = f(D) = 0$.*

1. *If all vertices A, B, C and D fall on a circle then*

$$\mathcal{D}(f, \mathcal{T}_1) = \mathcal{D}(f, \mathcal{T}_2).$$

2. *If $C \notin \omega(A)$ and $R(A) \leq \min(R(B), R(D))$ then*

$$\mathcal{D}(f, \mathcal{T}_1) < \mathcal{D}(f, \mathcal{T}_2).$$

PROOF: Let $\alpha = \frac{1}{2} f(A)$. Then

$$\mathcal{D}(f, \mathcal{T}_1) = \alpha^2 \left(\frac{|[B,C]|^2}{|ABC|} + \frac{|[C,D]|^2}{|ACD|}\right)$$

and

$$\mathcal{D}(f, \mathcal{T}_2) = \alpha^2 \, \frac{|[B,D]|^2}{|ABD|}.$$

By Lemma 7

$$|[A,C]| \cdot |[B,D]| \leq |[B,C]| \cdot |[A,D]| + |[A,B]| \cdot |[C,D]|.$$

Dividing by $|[A,B]| \cdot |[A,C]| \cdot |[A,D]|$ gives

$$\frac{|[B,D]|}{|[A,B]| \cdot |[A,D]|} \leq \frac{|[B,C]|}{|[A,B]| \cdot |[A,C]|} + \frac{|[C,D]|}{|[A,C]| \cdot |[A,D]|}$$

or

$$\frac{|[B,D]|^2}{|[A,B]|\cdot|[A,D]|\cdot|[B,D]|}$$
$$\leq \frac{|[B,C]|^2}{|[A,B]|\cdot|[A,C]|\cdot|[B,C]|}+\frac{|[C,D]|^2}{|[A,C]|\cdot|[A,D]|\cdot|[C,D]|}.$$

For any triangle it is well-known that the product of its sides equals the area times the radius of the circumcircle times 4. Hence

$$\frac{|[B,D]|^2}{|ABD|}\leq\frac{R(A)}{R(B)}\frac{|[B,C]|^2}{|ABC|}+\frac{R(A)}{R(D)}\frac{|[C,D]|^2}{|ACD|}$$

with equality if and only if the vertices fall on a circle. Consequently *1.* is immediate and for *2.* we notice

$$\frac{|[B,D]|^2}{|ABD|}<\frac{|[B,C]|^2}{|ABC|}+\frac{|[C,D]|^2}{|ACD|}$$

from the assumption $R(A)\leq\min(R(B),R(D))$. □

Lemma 13 *Let $ABCD$ be a strictly convex quadrilateral. Assume that the triangles ABD and BCD form a Delaunay triangulation $\mathcal{T}$. Let $\mathcal{T}^*$ be the triangulation ABC and ADC. Suppose that $f:\{A,B,C,D\}\to\mathbf{R}$. Then*

$$\mathcal{D}(f,\mathcal{T})\leq\mathcal{D}(f,\mathcal{T}^*).$$

If $\mathcal{D}(f,\mathcal{T})=\mathcal{D}(f,\mathcal{T}^*)$ then $f_{\mathcal{T}^*}$ is linear or the vertices fall on a circle.

Proof: We may assume that $R(A)\leq R(C)$. Let $h:\mathbf{R}^2\to\mathbf{R}$ be the linear function that equals f on $\{B,C,D\}$. Set $S=\{A,B,C,D\}$ and define $g:S\to\mathbf{R}$ by $g(A)=f(A)-h(A)$ and $g(B)=g(C)=g(D)=0$. Let p and q denote the piecewise linear extensions of g determined by $\mathcal{T}$ and $\mathcal{T}^*$ respectively. Then

$$\mathcal{D}(g,\mathcal{T})=\int\int_{ABCD}|\nabla p|^2\,dxdy\leq\mathcal{D}(g,\mathcal{T}^*)=\int\int_{ABCD}|\nabla q|^2\,dxdy$$

by the previous lemma. However

$$\mathcal{D}(f,\mathcal{T})-\mathcal{D}(g,\mathcal{T}^*)=\int\int_{ABCD}(|\nabla(p+h)|^2-|\nabla(q+h)|^2)\,dxdy$$
$$\leq 2\int\int_{ABCD}<\nabla(p-q),\nabla h>\,dxdy$$
$$=\int_{\partial(ABCD)}(p-q)\frac{\partial h}{\partial n}\,d\sigma-\int\int_{ABCD}(p-q)\Delta h\,dxdy=0$$

since h is harmonic in $\mathbf{R}^2$ and $p=q$ on the boundary of $ABCD$. □

PROOF: (Proof of Theorem 13) The theorem is a direct consequence of the flip algorithm. □

For a triangle T let $R(T)$ denote the radius of its circumcircle. For a triangulation $\mathcal{T}$ we now set

$$R_{\max}(\mathcal{T}) = \max\{R(T) : T \in \mathcal{T}\}$$

and

$$R_{\min}(\mathcal{T}) = \min\{R(T) : T \in \mathcal{T}\}.$$

We now have the interesting result that these functionals are minimized by Delaunay triangulations. For the functional $R_{\min}(\mathcal{T})$ see [3].

Theorem 14 *Let $\mathcal{T}$ be a Delaunay triangulation of the set S. Assume that $\mathcal{T}^*$ is a triangulation of the convex hull of S with vertex set S. Then*

$$R_{\min}(\mathcal{T}) \leq R_{\min}(\mathcal{T}^*) \tag{5}$$

and

$$R_{\max}(\mathcal{T}) \leq R_{\max}(\mathcal{T}^*). \tag{6}$$

PROOF: The theorem is a direct consequence of Lemma 9, Lemma 10 and the flip algorithm. □

For a triangulation $\mathcal{T}$ we let $R(\mathcal{T})$ denote the vector one gets by sorting the collection $\{R(T) : T \in \mathcal{T}\}$ in increasing order. Notice that $R_{\min}(\mathcal{T})$ and $R_{\max}(\mathcal{T})$ coincide with the first and last element in $R(\mathcal{T})$. We have the following sharpening of Theorem 14. If $\mathbf{x} = \{x_i\}_{i=1}^n$ and $\mathbf{y} = \{y_i\}_{i=1}^n$ are two sequences then we say that $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for $1 \leq i \leq n$.

Theorem 15 *Let $\mathcal{T}$ be a Delaunay triangulation of the set S. Assume that $\mathcal{T}^*$ is a triangulation of the convex hull of S with vertex set S. Then*

$$R(\mathcal{T}) \leq R(\mathcal{T}^*). \tag{7}$$

PROOF: Suppose $\mathcal{T}^{**}$ is derived from $\mathcal{T}^*$ by an edge swap of a strictly convex quadrilateral. Since $\{R(T) : T \in \mathcal{T}^{**}\}$ is derived from $\{R(T) : T \in \mathcal{T}^*\}$ by decreasing two elements it follows that $\{R(T) : T \in \mathcal{T}^{**}\} \leq \{R(T) : T \in \mathcal{T}^*\}$. The theorem is a direct consequence of the flip algorithm. □

Suppose $\Phi : \mathbf{R}^n \to \mathbf{R}$ is **symmetric**, i.e. for every permutation π of $\{1, \ldots, n\}$ we have for all $\mathbf{x} \in \mathbf{R}^n$

$$\Phi(x_1, \ldots, x_n) = \Phi(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

If $\mathcal{T}$ is a triangulation and if Φ is symmetric we set

$$\Phi_R(\mathcal{T}) = \Phi(R(T_1), \ldots, R(T_n)),$$

where $T_1, \ldots, T_n$ are the triangles in $\mathcal{T}$. Clearly $\Phi_R(\mathcal{T})$ is well-defined. We now let $\mathcal{A}_{\text{sym}}$ denote the class of all symmetric functions Φ such that if $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ then $\Phi(\mathbf{x}) \leq \Phi(\mathbf{y})$.

Theorem 16 *Let $\mathcal{T}$ be a Delaunay triangulation of the set S. Assume that $\mathcal{T}^*$ is a triangulation of the convex hull of S with vertex set S. Then*

$$\Phi_R(\mathcal{T}) \leq \Phi_R(\mathcal{T}^*) \tag{8}$$

for all $\Phi \in \mathcal{A}_{\text{sym}}$.

PROOF: The result is an obvious consequence from Theorem 15. □

Notice that $\max\{x_i\}$, $\min\{x_i\}$, $\Sigma_i x_i$, $\Sigma_i x_i^2$, $\Pi_i x_i$ and $\sqrt[n]{\Pi_i x_i}$ all belong to the class $\mathcal{A}_{\text{sym}}$.

We shall now study the area of piecewise linear surfaces. Let $\mathcal{T}$ be a triangulation with vertex set S. Assume $f : S \to \mathbf{R}$ and let $f_{\mathcal{T}}$ denote the piecewise linear extension to $\bigcup_{T \in \mathcal{T}}$ that is linear on each $T \in \mathcal{T}$. Set

$$\mathcal{A}(f, \mathcal{T}) = \Sigma_{T \in \mathcal{T}} \int\int_T \sqrt{1 + |\nabla f_{\mathcal{T}}|^2}\, dxdy.$$

Notice that $\mathcal{A}(f, \mathcal{T})$ is the area of the graph of $f_{\mathcal{T}}$.

Theorem 17 *Let $\mathcal{T}$ be a Delaunay triangulation of the set S. Let $f : S \to \mathbf{R}$ be any function. Then there is a positive number $\epsilon_0 = \epsilon_0(f)$ such that if $|\epsilon| < \epsilon_0$ and if $\mathcal{T}^*$ is a triangulation of the convex hull of S with vertex set S then*

$$\mathcal{A}(\epsilon f, \mathcal{T}) \leq \mathcal{A}(\epsilon f, \mathcal{T}^*).$$

To prove the theorem we need the following lemma.

Lemma 14 *Let $\mathcal{T}, \mathcal{T}^*$ be as above. If $\mathcal{D}(f, \mathcal{T}) = \mathcal{D}(f, \mathcal{T}^*)$ then $\mathcal{A}(f, \mathcal{T}) = \mathcal{D}(f, \mathcal{T}^*)$.*

PROOF: From Lemma 13 it follows that if we carry out an edge swap in a strictly convex quadrilateral Q in the flip algorithm then $f_{\mathcal{T}^*}$ must be linear in Q. Hence the area is unchanged which establishes the lemma. □

PROOF: (Proof of Theorem 17) We will argue by contradiction. Assume the conclusion of the theorem fails. Since there are only finitely many triangulations with vertex set S there exists $\epsilon_j \to 0$ such that

$$\mathcal{A}(\epsilon_j f, \mathcal{T}) > \mathcal{A}(\epsilon_j f, \mathcal{T}^*).$$

Since

$$\mathcal{A}(\epsilon f, \mathcal{T}) - \mathcal{A}(\epsilon f, \mathcal{T}^*) = \frac{1}{2}\epsilon^2(\mathcal{D}(f, \mathcal{T}) - \mathcal{D}(f, \mathcal{T}^*)) + O(\epsilon^4),$$

it follows that $\mathcal{D}(f, \mathcal{T}) \geq \mathcal{D}(f, \mathcal{T}^*)$. However from Theorem 13 we have $\mathcal{D}(f, \mathcal{T}) = \mathcal{D}(f, \mathcal{T}^*)$. Hence $\mathcal{A}(\epsilon f, \mathcal{T}) = \mathcal{A}(\epsilon f, \mathcal{T}^*)$ by Lemma 14. □

An interesting problem here is to construct a polynomial time algorithm for finding piecewise linear interpolating surface of least area.

For a triangle T let $\theta_{\min}(T)$ denote the smallest angle of T. If $\mathcal{T}$ is a triangulation we set

$$\theta_{\min}(\mathcal{T}) = \min\{\theta_{\min}(T) : T \in \mathcal{T}\}.$$

Lemma 15 *Let $ABCD$ be a strictly convex quadrilateral. Assume that the triangles ABD and BCD form a Delaunay triangulation $\mathcal{T}$. Let $\mathcal{T}^*$ be the triangulation ABC and ADC. Then*

$$\theta_{\min}(\mathcal{T}) \geq \theta_{\min}(\mathcal{T}^*)$$

with equality if and only if the vertices all fall on a circle.

PROOF: Let $\{\alpha_i\}_{i=1}^4$ and $\{\beta_i\}_{i=1}^4$ be the angles in $\mathcal{T}$ and $\mathcal{T}^*$ that are pairwise opposite the same side.

figure

Let a, b, c, d denote the interior angles of the polygon at the vertices A, B, C, D. Notice that $a = \beta_2 + \beta_4$ so $a > \theta_{\min}(\mathcal{T}^*)$. Similarly $c > \theta_{\min}(\mathcal{T}^*)$. If the vertices fall on a circle then $\alpha_i = \beta_i$ for $1 \leq i \leq 4$. Hence

$$\theta_{\min}(\mathcal{T}) = \theta_{\min}(\mathcal{T}^*)$$

in this case.

Assume now that the vertices do not fall on a circle. Set $\theta^* = \theta_{\min}(\mathcal{T})$. If $\theta^* \in \{a, c\}$ then we are done by the above inequalities. If $\theta^* \in \{\alpha_1, \ldots, \alpha_4\}$, say $\theta^* = \alpha_1$, we have the following. Let C^* be the intersection between the circumcircle to ABD and the line segment $[A, C]$. Let $\{\beta_i^*\}_{i=1}^4$ be the angles in the polygon ABC^*D that correspond to the β_i:s. Then $\beta_1^* > \beta_1$, so $\theta^* > \beta_1 \geq \theta_{\min}(\mathcal{T}^*)$. □

Theorem 18 *Let $\mathcal{T}$ be a Delaunay triangulation of the set S. Assume that $\mathcal{T}^*$ is a triangulation of the convex hull of S with vertex set S. Then*

$$\theta_{\min}(\mathcal{T}) \geq \theta_{\min}(\mathcal{T}^*).$$

Proof: The theorem is a direct consequence of Lemma 15 and the flip algorithm. □

A reference for the previous theorem is [34].

For a triangulation $\mathcal{T}$ we let $\Theta(\mathcal{T}) = \{\theta_i(\mathcal{T})\}$ denote the collection of all triangles in $\mathcal{T}$ sorted in increasing order. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two triangulations with the same number of triangles then we say that $\Theta(\mathcal{T}_2)$ is **lexicographically greater** than $\Theta(\mathcal{T}_1)$ if there is an integer m such that

$$\theta_j(\mathcal{T}_2) = \theta_j(\mathcal{T}_1) \text{ for } j < m$$

and

$$\theta_m(\mathcal{T}_2) > \theta_m(\mathcal{T}_1).$$

Corollary 1 *Let $\mathcal{T}, \mathcal{T}^*$ be two triangulations as in Theorem 18. If $\mathcal{T}^*$ is not a Delaunay triangulation then $\Theta(\mathcal{T})$ is lexicographically greater than $\Theta(\mathcal{T}^*)$.*

PROOF: It follows from Lemma 15 that the flip algorithm makes the angle sequence lexicographically greater. □

A reference for the previous result is [35].

4.4 Farthest Delaunay Triangulations

A farthest Delaunay triangulation is characterized by the property that the circumcircle of each triangle contains the vertex set of the triangulation. This is only possible if the vertices are the extreme points of their convex hull. More precisely we have the following definition.

Definition 1 *Assume that S is the set of vertices of a strictly convex polygon. Let $\mathcal{T}$ be a triangulation of* $\mathrm{conv}(S)$ *with vertex set S. $\mathcal{T}$ is called a* **farthest Delaunay triangulation** *is $S \subset \Delta(T)$ for all $T \in \mathcal{T}$, where $\Delta(T)$ denotes the closed disk whose boundary contains the vertices of T.*

Let $f(x) = |x|^2$, $x \in \mathbf{R}^2$, and let $S_f = \{(p, f(p)) : p \in S\}$. Let $\{F_i\}$ be the faces of the upper part of the convex hull of S_f. Triangulating the projection of each of the faces F_i yields a farthest Delaunay triangulation.

Let R be a strictly convex quadrilateral and assume that the vertices of R do not lie on a circle. Then R has exactly two triangulations — one being the Delaunay triangulation and the other the farthest Delaunay triangulation.

We shall now describe the analogue of the flip algorithm for farthest Delaunay triangulations.

Let $\mathcal{T}$ be a triangulation and assume that e is the diagonal of a strictly convex quadrilateral R formed by two triangles in $\mathcal{T}$ with a common edge e. Assume that the vertices of R are not cocircular. The **reversed flip algorithm** consists of making a diagonal swap to obtain the farthest Delaunay triangulation of R. The flip algorithm consists of selecting the Delaunay triangulation of R.

Theorem 19 *Let S be the vertex set of a strictly convex polygon and let $\mathcal{T}$ be any triangulation of* $\mathrm{conv}(S)$ *with vertex set S. Applying the reversed flip algorithm to $\mathcal{T}$ terminates in a farthest Delaunay triangulation of S.*

PROOF: Let $R(\mathcal{T})$ denote the radii of the circumcircles of the triangles in $\mathcal{T}$ sorted in increasing order. If $\mathcal{T}'$ is a triangulation obtained by applying

the reversed flip algorithm then $R(\mathcal{T}) \leq R(\mathcal{T}')$ and at least two components differ. Hence the algorithm must terminate.

We shall now prove that if the reversed flip algorithm has terminated with a triangulation $\mathcal{T}^*$ then $\mathcal{T}^*$ is a farthest Delaunay triangulation. Let N be the number of elements in S. The result is obvious if $N \leq 4$. For T a triangle let $\Delta(T)$ be the closed disk whose boundary contains the vertices of T. Suppose that there exists a $T_0 \in \mathcal{T}^*$ such that for some $p \in S$ we have $p \notin \Delta(T_0)$. Let e be an edge of T_0 that separates p from T_0. Let H be the closed half plane determined by e that contains p. Set $S^* = S \bigcap H$ and $\mathcal{T}^{**} = \{T \in \mathcal{T}^* : T \subset H\}$. Since S^* has fewer elements than S it follows by the induction assumption that $\mathcal{T}^{**}$ is a farthest Delaunay triangulation of S^*. Let $T_1 \in \mathcal{T}^{**}$ be the triangle that has e as an edge. Then $R = T_0 \bigcup T_1$ is strictly convex. Since $p \in \Delta(T_1)$ but $p \notin \Delta(T_0)$ we have that the vertices of R are not on a circle. Also $T_0 \subset \Delta(T_1)$ and $\mathrm{Int}(T_0) \bigcap H = \emptyset$. This is impossible. □

Using our previous notation we have the following properties of farthest Delaunay triangulations.

Theorem 20 *Let S be the vertex set of a strictly convex polygon. Let $a, b \in S$, $a \neq b$, and assume there is a closed disk B such that $S \subset B$ and $a, b \in \partial B$. Then there is a farthest Delaunay triangulation $\mathcal{T}$ of S such that $[a, b]$ is an edge of $\mathcal{T}$. Assume there is a closed disk Δ such that $S \subset \Delta$ and $a, b, c \in \partial\Delta$. Then the triangle with vertices a, b, c is a triangle in some farthest Delaunay triangulation of S.*

Theorem 21 *Let S be the vertex set of a strictly convex polygon. Let $\mathcal{T}$ be a farthest Delaunay triangulation of S. Let $\mathcal{T}^*$ be any triangulation of* $\mathrm{conv}(S)$ *with vertex set S. Then the following holds:*

1. $R_{\max}(\mathcal{T}^*) \leq R_{\max}(\mathcal{T})$.
2. $R_{\min}(\mathcal{T}^*) \leq R_{\min}(\mathcal{T})$.
3. $R(\mathcal{T}^*) \leq R(\mathcal{T})$.
4. *If $\Phi \in \mathcal{A}_{\mathrm{sym}}$ then*
$$\Phi_R(\mathcal{T}^*) \leq \Phi_R(\mathcal{T}).$$
5. *If $f : S \to \mathbf{R}$ is any function then*
$$\mathcal{D}(f, \mathcal{T}^*) \leq \mathcal{D}(f, \mathcal{T}).$$

6. $\theta_{\min}(\mathcal{T}) \leq \theta_{\min}(\mathcal{T}^*)$.

7. $\Theta(\mathcal{T}^*)$ *is lexicographically greater than* $\Theta(\mathcal{T})$.

4.5 General Properties of Voronoi Diagrams

The Voronoi diagram is a basic structure for the understanding of point sets in a metric space. It is a natural tool for analyzing the geometric distribution and internal relations of data points. Here is the general definition. Let S be a finite set of points in a metric space X with metric δ. Set

$$V_S(p) = \{x \in X : \delta(x,p) \leq \min\{\delta(x,q) : q \in S \setminus \{p\}\}\},$$

i.e. $V_S(p)$ is the set of points x that are not closer to any other point in S than to p. We call $V_S(p)$ the **Voronoi cell** associated to p.

Proposition 3 *The Voronoi cells have the following properties:*

1. $p \in V_S(p)$ *for all* $p \in S$

2. $\bigcup_{p\in S} V_S(p) = X$

3. *If* $x \in V_S(p) \bigcap V_S(q)$ *then* $\delta(x,p) = \delta(x,q)$.

We call the collection of all Voronoi cells the **Voronoi diagram** of S. As an indication of its usefulness, the Voronoi diagrams has independently been "discovered" in many fields, for a discussion of this see [2] and [40], in particular pp 6 – 10 in [40].

There are several natural extensions. For instance, we can associate to every subset $Y \subset S$, $Y \neq \emptyset$, the **generalized Voronoi cell** $V_S(Y)$ by letting

$$V_S(Y) = \{x \in X : \text{for all } p \in Y \text{ and all } q \in S \setminus Y \ \ \delta(x,p) \leq \delta(x,q)\}.$$

That is $V_S(Y)$ is the locus of all points x such that each point in Y is at least as close to x as to any point in $S \setminus Y$. Of course, it may happen that $V_S(Y)$ is empty. The Voronoi diagram is of **order** k if it is the collection of all (non-empty) generalized Voronoi cells of subsets of S with k elements.

Recently, there has been an interest in a still more general construction. Let $E_1, E_2, \ldots, E_N$ be N closed subsets of X and let

$$d_k(x) = \inf\{\delta(x,y) : y \in E_k\}.$$

We consider the cells defined by

$$\{x \in X : d_k(x) \leq \min\{d_j(x) : j \neq k\}\}.$$

For more about this see [40].

4.6 Classical Voronoi Diagrams in the Euclidean Plane

We shall in this section let $S \subset \mathbf{R}^2$ be a finite set of points that is not included in any line. In particular, S has at least three elements. In this case the Voronoi cell

$$V_S(p) = \{x \in \mathbf{R}^2 : |x - p| \leq \min\{|x - q| : q \in S \setminus \{p\}\}\}$$

is a possibly unbounded polygonal domain. Therefore $V_S(p)$ is often called the **Voronoi polygon** associated to p.

Let p and q be two distinct points in $\mathbf{R}^2$ and let $b(p, q)$ be the line that perpendicularly bisects the line segment $[p, q]$. We call $b(p, q)$ the **bisector** to p and q. The bisector divides the plane into two half planes and we denote by $H(p, q)$ that closed half plane that contains p.

Proposition 4 *For all $p \in S$ we have that*

$$V_S(p) = \bigcap_{q \in S \setminus \{p\}} H(p, q),$$

and

$$\bigcup_{p \in S} V_S(p) = \mathbf{R}^2.$$

If $p, q \in S$, $p \neq q$, then

$$\mathrm{Int}(V_S(p)) \bigcap \mathrm{Int}(V_S(q)) = \emptyset.$$

$V_S(p)$ is a closed convex polygonal domain with non-empty interior. Moreover, $V_S(p)$ is unbounded if and only if $p \in S \bigcap \partial \mathrm{conv}(S)$.

Proof: If $p \in S \bigcap \mathrm{Int}(\mathrm{conv}(S))$ then there is a triangle T with vertices in $\mathrm{ext}(S)$ such that $p \in \mathrm{Int}(T)$. It is easily seen that

$$V^* = \bigcap_{q \in \mathrm{ext}(T)} H(p, q)$$

is bounded. Hence $V_S(p)$ is bounded in this case, which easily yields the proposition. □

We call the edges of the Voronoi polygons for **Voronoi edges**. We will say that a Voronoi edge e is **generated** by $p, q \in S$ if $e = V_S(p) \bigcap V_S(q)$. The vertices of the Voronoi polygons are called **Voronoi vertices**. The following proposition is an immediate consequence of Proposition 4.

Proposition 5 *All Voronoi edges are half lines or bounded line segments. Let the Voronoi edge e be generated by p and q. Then e is bounded if and only if*

$$[p, q] \bigcap \mathrm{Int}(\mathrm{conv}(S)) \neq \emptyset.$$

Moreover, e is unbounded if and only if

$$[p, q] \subset \partial(\mathrm{conv}(S)).$$

Proposition 6 *Let $p \in S$ and let $q \in S \setminus \{p\}$ be the point in $S \setminus \{p\}$ that is closest to p. Then p and q generate a Voronoi edge.*

PROOF: Let $\eta = \dfrac{p+q}{2}$. It is enough to show that $[p, \eta]$ does not intersect any Voronoi edge generated by p and some $w \in S \setminus \{p, q\}$. If $\eta_m \in [p, \eta]$ belongs to the Voronoi edge generated by p and some $w \in S \setminus \{p, q\}$ then

$$\frac{|p - w|}{2} < |p - \eta_m| \leq |p - \eta| = \frac{|p - q|}{2}.$$

Since q was closest to p we must have $|p - w| = |p - q| = |p - \eta_m|$. Hence $\eta = \eta_m$, which is impossible because $w \neq q$. □

Let Q_S be the collection of all Voronoi vertices for S. Since the Voronoi polygons are convex, at least three Voronoi edges meet at any Voronoi vertex.

Proposition 7 *For $q \in Q_S$ let Δ_q be the maximal closed disk Δ centered at q such that $\mathrm{Int}(\Delta) \bigcap S = \emptyset$. Then $\partial\Delta_q \bigcap S$ has at least three elements.*

PROOF: For $q \in Q_S$ let

$$V^*(q) = \{p \in S : V_S(p) \text{ has a vertex at } q\}.$$

If $p_1, p_2 \in V^*(q)$ share an edge then $|q - p_1| = |q - p_2|$. Let r be the common value of $\{|p - q| : p \in V^*(q)\}$. Pick $p_0 \in \partial\Delta_q \bigcap S$. From the definition of $V_S(p_0)$ follows that $q \in V_S(p_0)$, i.e. $p_0 \in V^*(q)$. Hence $\partial\Delta_q \bigcap S = V^*(q)$ and since $V^*(q)$ has at least three elements we are done. □

For $q \in Q_S$ let $V_S^*(q)$ be the points $p \in S$ for which $V_S(p)$ has a vertex at q. We denote by $W_S(q)$ the closed convex hull of $V_S^*(q)$.

Theorem 22 *We have that*

$$\bigcup_{q \in Q_S} W_S(q) = \mathrm{conv}(S). \tag{9}$$

If $q_1, q_2 \in Q_S$ with $q_1 \neq q_2$ then

$$\mathrm{Int}(W_S(q_1)) \bigcap \mathrm{Int}(W_S(q_2)) = \emptyset. \tag{10}$$

Moreover, $V_S^*(q)$ is contained in a circle centered at q.

PROOF: Let Δ_q be the maximal disk centered at q for which $\mathrm{Int}(\Delta_q) \bigcap S = \emptyset$. If $q \in Q_S$ then $V_S^*(q) \subset \partial\Delta_q$ by Proposition 7. Hence $W_S(q) \subset \Delta_q \bigcap \mathrm{conv}(S)$.

We shall now establish (10). Let $q_1, q_2 \in Q_S$ with $q_1 \neq q_2$. If $\mathrm{Int}(\Delta_{q_1}) \bigcap \mathrm{Int}(\Delta_{q_2}) = \emptyset$ then (10) follows trivially. Assume now that

$$\mathrm{Int}(\Delta_{q_1}) \bigcap \mathrm{Int}(\Delta_{q_2}) \neq \emptyset.$$

Neither Δ_{q_1} nor Δ_{q_2} is contained in the other. Hence $\partial\Delta_{q_1} \bigcap \partial\Delta_{q_2}$ consists of two points z_1 and z_2. Let l be the line through z_1 and z_2. Then l separates q_1 and q_2 and let H_i be the closed half plane determined by l such that $q_i \in H_i$. Clearly $\mathrm{Int}(H_1) \bigcap \mathrm{Int}(H_2) = \emptyset$. Clearly $V_S^*(q_1) \bigcap \mathrm{Int}(\Delta_{q_2}) = \emptyset$. Since $V_S^*(q_1) \subset \Delta_{q_1}$ if follows that

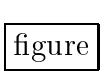
$$V_S^*(q_1) \subset \Delta_{q_1} \setminus (\mathrm{Int}\Delta_{q_2}) \subset H_1.$$

Similarly

$$V_S^*(q_2) \subset H_2.$$

Hence $W_S(q_1) \subset H_1$ and $W_S(q_2) \subset H_2$, which yields (10).

Let $X = \bigcup_{q \in Q_S} W_S(q)$. Clearly $X \subset \mathrm{conv}(S)$ but assume $X \neq \mathrm{conv}(S)$. Hence there is a non-empty disk B such that $B \subset \mathrm{conv}(S)$ but $X \bigcap X = \emptyset$. Let l be any line such that $l \bigcap B \neq \emptyset$, $l \bigcap S = \emptyset$ but $l \bigcap \mathrm{Int}(W_S(v)) \neq \emptyset$ for some $v \in Q_S$. Let x be any point in $B \bigcap l$ and let $w \in l \bigcap X$ be a point that is closest to x. Then $w \in W_S(q_1)$ for some $q_1 \in Q_S$. In addition, $l \bigcap \mathrm{Int}(W_S(q_1)) \neq \emptyset$ and w belongs to a unique edge $[p_1, p_2]$ of $W_S(q_1)$. Of course $p_1, p_2 \in S$.



Let e be the Voronoi edge that crosses $[p_1, p_2]$. Then q_1 is an end point of e. Furthermore, e is generated by p_1 and p_2. Let l_1 be the line through p_1 and p_2. Let $p_3 \in V_S^*(q_1)$ be different from both p_1 and p_2. Then l_1 separates x and p_3 so $[p_1, p_2] \bigcap \mathrm{Int}(\mathrm{conv}(S)) \neq \emptyset$. From Proposition 7 we have that e must be bounded. Let q_2 be the end point of e that is different from q_1. Now $p_1, p_2 \in V_S^*(q_2)$. Hence $w \in \mathrm{Int}(W_S(q_1) \bigcap W_S(q_2))$, so $w \in \mathrm{Int}X$. This contradicts the definition of w, which yields the theorem. □

Triangulating each $W_S(q)$ gives now a Delaunay triangulation.

Corollary 2 *Let $\mathcal{T}$ be a triangulation of* $\mathrm{conv}(S)$ *with vertex set S such that each $T \in \mathcal{T}$ is contained in some $W_S(q)$, $q \in Q_S$. Then $\mathcal{T}$ is a Delaunay triangulation for S.*

Proof: Immediate consequence of Theorem 22. □

For $p \in \mathbf{R}^2$ let h_p be the half space

$$\{(x, y) : x \in \mathbf{R}^2, y \in \mathbf{R} \text{ and } y \geq 2 < x, p > -|p|^2\}.$$

Then the boundary of h_p is the tangent plane to the paraboloid $y = |x|^2$ at $x = p$. Let $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ be the projection that sends (x, y) into x. An easy computation shows that $x \in \pi(\partial h_p \bigcap h_q)$ if and only if $|x - p| \leq |x - q|$. Hence we have the following method for constructing the Voronoi diagram in $\mathbf{R}^2$.

Proposition 8 *Let h_S be the polyhedron*

$$h_S = \bigcap_{p \in S} h_p.$$

Then the Voronoi polygons for S are the projections of the faces of h_S.

4.7 Pattern Recognition and Subgraphs of Delaunay Triangulations

For pattern recognition one has developed several graphs associated to a point set. We will in this section describe some of these and their relationships to Delaunay triangulations.

Throughout this section we let S denote a finite point set in $\mathbf{R}^2$ that is not contained in a line. In particular, there are at least three points in S.

Setting $f(x) = |x|^2$, $x \in \mathbf{R}^2$, we let $S_f = \{(p, f(p)) : p \in S\}$ be the lift of S. Denote by L_S the lower part of the convex hull of S_f, see section 4.1. We let $DG(S)$ be the **Delaunay graph** of S, i.e. the graph with vertex set S and pq is an edge in $DG(S)$ if and only if $(p, f(p))$ and $(q, f(q))$ are the end points of an edge of L_S. Notice that $DG(S)$ is a subgraph of the skeleton of any Delaunay triangulation.

Let S be the vertex set of a convex polygon and let U_S denote the upper part of the convex hull of S_f. We denote by $FDG(S)$ the **farthest Delaunay graph**, i.e. the graph with vertex set S and $p, q \in S$ form an edge if and only if $(p, f(p))$ and $(q, f(q))$ form an edge in U_S. For a general set S we let $FDG(S)$ denote $FDG(\text{ext}(S))$.

The Gabriel graph, see [23], [37] and [58], is used for defining adjacency in point patterns. If $p, q \in S$, $p \neq q$, then pq is an edge in the **Gabriel graph** $GG(S)$ if and only if

$$\Delta_{pq} \bigcap (S \setminus \{p, q\}) = \emptyset,$$

where Δ_{pq} is the closed disk that has $[p, q]$ as a diameter.

Theorem 23 *The Gabriel graph $GG(S)$ of S is a subgraph of the Delaunay graph $DG(S)$ of S.*

PROOF: From Theorem 10 follows that every edge in $GG(S)$ is the edge of some triangle in some Delaunay triangulation of S. Assume that pq is an edge in $GG(S)$ but pq is not an edge in $DG(S)$. Let T be a triangle in a Delaunay triangulation of S such that $[p, q]$ is an edge of T. Let Δ be the closest disk whose boundary contains the vertices of T. Then $\text{Int}(\Delta) \bigcap S = \emptyset$ and $\partial\Delta \bigcap (S \setminus \{p, q\}) \neq \emptyset$. hence $[p, q]$ is not a diameter of S. Let h_+, h_- be the two open half planes determined by the line through p and q. Since $[p, q]$ does not belong to $DG(S)$ we must have $\Delta \bigcap S \bigcap h_+ \neq \emptyset$ and $\Delta \bigcap S \bigcap h_- \neq \emptyset$. Since $\Delta \neq \Delta_{pq}$ we must have that $\Delta \bigcap h_+$ or $\Delta \bigcap h_-$ is contained in $\text{Int}(\Delta_{pq})$. This contradiction establishes the theorem. □

If $p \neq q$ we let L_{pq} denote the **lune**

$$L_{pq} = \{w \in \mathbf{R}^2 : |w - p| \leq r\} \bigcap \{w \in \mathbf{R}^2 : |w - q| \leq r\},$$

where $r = |p - q|$.

We can now define the **relative neighbourhood graph** $RNG(S)$ of the set S. For reference see [57] and [56]. If $p, q \in S$ then pq is an edge of $RNG(S)$ if and only if

$$\mathrm{Int}(L_{pq}) \bigcap S = \emptyset.$$

Since $\Delta_{pq} \subset L_{pq}$ and $\Delta_{pq} \setminus \{p, q\} \subset \mathrm{Int}(L_{pq})$ we have the following result.

Theorem 24 *The relative neighbourhood graph $RNG(S)$ is a subgraph of the Gabriel graph $GG(S)$.*

A **Euclidean minimum spanning tree** of S, the set of such graphs is denoted by $EMST(S)$, is a tree having the vertex set S and for which the sum of the length of all edges attains the minimum over all trees having the vertex set S. See [25] and [48].

Theorem 25 *An edge of an Euclidean minimum spanning tree for S is an edge of $RNG(S)$.*

PROOF: We will argue by contradiction. Assume pq is an edge of a Euclidean spanning tree G that is not an edge of $RNG(S)$. Removing the edge pq from G splits G into two subtrees G_1 and G_2, say p is in G_1 and q is in G_2. Our assumption means that there is a $w \in \mathrm{Int}(L_{pq}) \bigcap S$. We may without loss of generality assume that $w \in G_1$. We now get a connected graph G_3 by taking the union of G_1 and G_2 with the edge wq. Since the total length of the edges in G_3 is smaller than the total length of the edges in G we get a contradiction. □

The above subgraphs of the Delaunay triangulation are all undirected. We shall now introduce a directed graph of S, the **nearest neighbour graph** $NNG(S)$, by letting $\vec{pq}$, $p \neq q$, be an edge if and only if

$$|p - q| = \min\{|p - w| : w \in S \setminus \{p\}\}.$$

For reference see [33], [50] and [43].

Theorem 26 *Let $p, q \in S$, $p \neq q$. If $\vec{pq}$ is an edge in $NNG(S)$ then pq is an edge in $RNG(S)$. If there is no other edge in $NNG(S)$ having initial vertex p then pq is an edge of every $EMST(S)$.*

PROOF: The first statement is obvious. We shall now prove the second part. Assume to the contrary that pq is not an edge of a Euclidean spanning tree G. Let G_1 be the graph one gets by adding pq as an edge to G. Then G_1 has a cycle Γ that contains pq. Hence there is a $w \in S \setminus \{p, q\}$ such that pw is an edge of G that is on Γ. Let G_2 be the graph one gets by removing the edge pw from G_1. Then G_2 is a spanning tree for S. Since the edge pq is shorter than pw we have that the total length of the edges in G_2 is smaller than that of G. We get a contradiction. □

We next quote a remarkable result from [29].

Theorem 27 *Let $\mathcal{T}$ be a Delaunay triangulation of S. For $p, q \in S$ let $D(p, q)$ denote the length of the shortest chain of edges of $\mathcal{T}$ connecting p and q. Then*

$$D(p, q) \leq C|p - q|,$$

where $C = \dfrac{2\pi}{3\cos(\frac{\pi}{6})} \approx 2.42$.

For a proof see [29].

We shall now describe a generalization of the convex hull of S. This generalization leads to a one-parametric family of subdivisions which intuitively connects the "crude" and "fine" shape of S.

The generalization of the convex hull is based on the notion of α–**disks** with α any real number. This is

1. closed disks of radii $\dfrac{1}{\alpha}$ if $\alpha > 0$;
2. closed half planes if $\alpha = 0$;
3. the complements of closed disks of radii $-\dfrac{1}{\alpha}$ if $\alpha < 0$.

We now define the α–**hull** of S as the intersection of all α–disks that contain S. We denote the α–hull of S by $\text{Hull}_\alpha(S)$. We will use the convention that $\text{Hull}_\alpha(S)$ is the entire plane if S is not contained in any α–disk. We have the following result.

Proposition 9 *If* $\alpha_1 < \alpha_2$ *then* $\text{Hull}_{\alpha_1}(S) \subset \text{Hull}_{\alpha_2}(S)$.

PROOF: Assume A_2 is an α_2–disk that contains S. Since A_2 is the intersection of all α_1–disks that contain A_2 we have that

$$\text{Hull}_{\alpha_1}(S) \subset A_2.$$

Taking the intersection of all α_2–disks that contain S yields the proposition. □

We shall now introduce the notion of α-shape. A general reference for α-shape is [20] and [19] chapter 13.2.6.

We say that a point $p \in S$ is α-**extreme** if for all sufficiently small $\epsilon > 0$ there is an $(\alpha + \epsilon)$-disk $B_{\alpha+\epsilon}$ such that $S \subset B_{\alpha+\epsilon}$ and $S \bigcap \partial B_{\alpha+\epsilon} = \{p\}$. Assume now that $p, q \in S$, $p \neq q$. We will say that an α-disk B_α **leans on** the ordered pair (p, q) if $p, q \in \partial B_\alpha$ and an infinitesimal movement of B_α normal to and towards the left of the vector $q - p$ puts p and q outside B_α.

figure

Furthermore the ordered pair (p, q) is said to be α-**exposed** if for all sufficiently small $\epsilon > 0$ there is an $(\alpha + \epsilon)$-disk $B_{\alpha+\epsilon}$ such that $S \subset B_{\alpha+\epsilon}$ and $B_{\alpha+\epsilon}$ leans on (p, q).

Let l_{pq} denote the line through p and q that is directed from p to q. We note that (p, q) is 0-extreme if and only if $[p, q]$ is an edge of $\text{conv}(S)$ and no point of S lies to the right of l_{pq}.

figure

Proposition 10 *Suppose $p, q \in S$, $p \neq q$. Then the following holds:*

1. *There is a real number A_p such that p is α-extreme if and only if $\alpha < A_p$.*

2. *Suppose (p, q) is α-exposed for some α. Then there are real numbers A_{pq} and B_{pq} such that (p, q) is α-exposed if and only if $A_{pq} \leq \alpha < B_{pq}$.*

3. *If (p, q) is α-exposed then both p and q are α-extreme.*

4. *If $\alpha \geq 0$ and p is α-extreme then there is a $w \in S \setminus \{p\}$ such that (p, w) is α-exposed.*

PROOF: We now give the arguments for the statements in the order they are stated.

1. Let I_p be the set of all α such that p is α-extreme. Clearly I_p is bounded from above and $I_p \neq \emptyset$. By elementary geometry we see that if $\beta \in I_p$, then there is an $\epsilon > 0$ such that $(-\infty, \beta + \epsilon) \subset I_p$.

2. Let I_{pq} be the set of all α such that (p, q) is α-exposed. By assumption $I_{pq} \neq \emptyset$. Clearly I_{pq} is a bounded set. Let I^*_{pq} be the set of $\rho \in \mathbf{R}$ such that there is a ρ–disk B_ρ with $S \subset B_\rho$ and B_ρ leans on (p, q). Assume $\alpha_1 \leq \alpha_2$ and $\alpha_1, \alpha_2 \in I_{pq}$. By elementary geometry we have that if $\epsilon > 0$ is small enough then $[\alpha_1, \alpha_2 + \epsilon)$ is contained in I_{pq}. Hence I_{pq} is an interval. Let $A_{pq} = \inf I_{pq}$ and $B_{pq} = \sup I_{pq}$. Clearly $(A_{pq}, B_{pq}) \subset I^*_{pq}$. Hence $A_{pq} \in I_{pq}$ so $I_{pq} = [A_{pq}, B_{pq})$.

3. This is an immediate consequence of the definition of α–exposed.

4. Since the result is obvious for $\alpha = 0$ we will assume $\alpha > 0$. Let B be the smallest closed disk such that $S \subset B$ and $p \in \partial B$. Pick $p^* \in \partial B$ such that $[p, p^*]$ is a diameter of B. Let λ be the ray that originates in p and passes through p^*. Let Q be the circle with radius $\dfrac{1}{\alpha}$ and center p. If ρ denotes the radius of B then $\rho < \dfrac{1}{\alpha}$. Let p_0 be the intersection between Q and λ. Let Δ_q be the disk centered at q with radius $\dfrac{1}{\alpha}$. Notice that $B \subset \Delta_{p^*}$. In particular, $S \subset \Delta_{p^*}$ and $(S \setminus \{p\}) \bigcap \partial\Delta_{p^*} = \emptyset$.

figure

We have that $p \in \partial\Delta_q$ for all $q \in Q$. Starting at $q = p^*$ we now move q in the counter clockwise direction along Q until $(S \setminus \{p\}) \bigcap \partial\Delta_q \neq \emptyset$. Again let q denote this point on Q and set $E = (S \setminus \{p\}) \bigcap \Delta_q$. Let $w \in E$ be the furthest from p. Since $S \subset \Delta_q$ it is easily seen that (p, w) is α–exposed.

□

Let S_α be the points $p \in S$ such that for some $q \in S$ we have that (p, q) or (q, p) is α–exposed. We define the α–graph $EG_\alpha(S)$ as the graph on S_α such that $p, q \in S_\alpha$ form an edge if and only if (p, q) or (q, p) is α–exposed.

Theorem 28 *Let $\alpha \in \mathbf{R}$ and assume e is an edge in $EG_\alpha(S)$. If $\alpha \geq 0$ then e is an edge in $FDG(S)$. If $\alpha < 0$ then e is an edge of $DG(S)$.*

PROOF: This is an immediate consequence of Theorem 20 and Theorem 10.
□

5 Arrangements

Arrangements form one of the important structures in computational geometry. We slightly touch on the notion of duality and use results on Schinzel-Davenport sequences to prove the zone theorem.

5.1 Duality

We study two applications of geometric duality. The first one yields a fast algorithm for computing the minimum area triangle from among n points in the plane, see [12]. The second one gives a preprocessing algorithm for answering "ray-shooting" queries quickly, see [11].

Min-Area Triangles

To find the minimum area triangle, we first focus our attention on the problem of finding a min area triangle that has two fixed points P_i and P_j chosen as the "base" of the triangle. Our task reduces to finding for each pair of points, P_i and P_j, the point P_k whose normal distance to the line through P_i and P_j is the least. Say the line ℓ through P_iP_j has slope m and y intercept c. For all lines having slope m, through other points, the line through P_k has y intercept nearest to c. Call its intercept c'. (This follows from the fact that the normal distance between the lines is proportional to their y intercepts.)

In the dual, this line will be the first point vertically below (above) the point of intersection of the duals of P_i and P_j. This intersection point will have coordinates, (m, c'), and the line closest to this, and vertically below (above), is the dual of P_k, passing through the point (m, c').

To compute point P_k, for every pair of points P_iP_j, we construct the arrangement of the dual graph. While constructing the arrangement, we keep track of which line is immediately below which point. This can can easily be done while constructing the arrangement itself. (Recall that in the proof of the zone theorem, we do a "walk" around each face that the new line is incident to. While doing this weak we can perform the required updates.) This gives us all the information we need. To compute the min area triangle, we minimize over all P_i and P_j, which can be done in $O(n^2)$ time, where n is the number of points.

Figure

Ray-Shooting

Given a simple polygon P of n vertices, we wish to preprocess this polygon to answer ray-shooting queries of the following form: given a query point $q \in P$ and a direction $\vec{d}$, which is the *first* edge of the polygon that is hit when we shoot a ray from q? We will show that these queries can be answered in $O(\log^2 n)$ time. (Chazelle and Guibas provide a more complex $O(\log n)$ query time algorithm as well.)

The main idea is to decompose the polygon into a "hierarchy" of polygons. We use the "polygon-cutting theorem" of Chazelle's to do this. The polygon cutting theorem states that every simple polygon with n vertices ($n > 3$) contains a diagonal such that adding the diagonal, partitions the polygon into two polygons P_1 and P_2, with each P_i having at most $\frac{2}{3}n$ vertices.

We now construct a tree using this decomposition. The vertices of this tree correspond to polygons. The root corresponds to the polygon P. We apply the polygon cutting theorem to P, and the children of this vertex correspond to polygons P_1 and P_2 respectively. We now apply the polygon cutting theorem recursively to each node, until the corresponding polygon is a triangle. Observe that the height of this tree is $O(\log n)$, since we are reducing the size of the polygons by a constant factor at each step. For any polygon P, let $P(e)$ denote the diagonal that is used to decompose the polygon. We assume that each node in the tree has the diagonal information available.

The polygon cutting theorem can be proved using the fact that the dual of the triangulation is a tree, and we can prove that each such tree (vertices with degree at most 3) always contains an edge that partitions the tree into smaller trees with at most $\frac{2}{3}n$ vertices.

We solve the query $hit(P, q, \vec{d})$ as follows. We assume that $q \in P$. If $q \in P_1$ then the only way the ray can exit the polygon from δP_2 is if it crosses edge $P(e)$ without crossing any polygon boundary of P_1.

Figure

Step 1. (Assume that $q \in P_1$.) If the ray from point q with direction $\vec{d}$ does not intersect edge $P(e)$, then recursively solve $hit(P_1, q, \vec{d})$.

Step 2. If the ray appears to pierce $p(e)$ at point p, then we have to check if

the ray is able to reach $P(e)$ from q. If so, we recursively solve $hit(P_2, p, \vec{d})$ else we recursively solve $hit(P_1, q, \vec{d})$.

We still need to check if the ray from q reaches $P(e)$. We do this by shooting a ray from p towards q. If the ray reaches q, then the ray from q must reach $P(e)$. Compute $o = reach(P_1, p, -\vec{d})$ – if the segment $\overline{po}$ contains q then the ray from q in direction d reaches p, otherwise it crosses the boundary at an earlier point. This reduces the problem to another ray shooting query. Why is this query any easier to solve?

We will show that such queries are simpler, since they are being shot from a particular edge on the boundary of P_1, namely the edge $P(e)$. Chazelle and Guibas make use of a duality transformation that establishes a correspondence between points (on a two sided plane) and rays.

Any ray r_i shot from $P(e)$ becomes a point p_i in dual space. We decompose the dual space into "regions" with the property that if two rays r_1 and r_2 from $P(e)$ hit the same edge e' (perhaps at different places) then the points p_1 and p_2 are in the same "region". Moreover, they prove that this region is *convex*. The main idea is that any query ray r' shot from edge $P(e)$ can be transformed into a point p', and by doing a *point-location* query in $O(\log n)$ time, we can determine the edge that ray r' hits (based on the region it falls into). Once we know the edge, it is easy to compute the intersection point. This takes $O(\log n)$ time.

The entire algorithm takes $O(\log^2 n)$ time, since we start at the root of the polygon tree, and proceed down the tree doing a point location query at each level. The tree has $O(\log n)$ levels, and each point location takes $O(\log n)$ time. Further data structure tricks are used to reduce the query time to $O(\log n)$.

5.2 Davenport-Schinzel Sequences

A Davenport-Schinzel sequence is a sequence $U = (u_1, \ldots, u_m)$ constructed over an alphabet of size n. We define a **Davenport-Schinzel sequence** of order s as follows (referred to as $DS(n, s)$). There are two conditions that these sequences are required to satisfy.

1. $u_i \neq u_{i+1}$ for each $i < m$.
2. There do not exist $s+2$ indices $1 \leq i_1 < i_2 < \ldots < i_{s+2} \leq m$ such that

$$u_{i_1} = u_{i_3} = u_{i_5} = \ldots = a$$

$$u_{i_2} = u_{i_4} = u_{i_6} = \ldots = b$$

and $a \neq b$.

This means that the presence of long alternations of any pair of distinct symbols in Davenport-Schinzel sequences are forbidden. As an example we see that 122341 is not a DS sequence since 2 occurs consecutively, and that 14234132 is a DS sequence of order 3, but not of order 2, since there is an alternating subsequence $1 \ldots 2 \ldots 1 \ldots 2$.

Let $\lambda_s(n)$ be the length of a longest $DS(n, s)$ sequence. We are interested in deriving good upper bounds on $\lambda_s(n)$, as function of n. These sequences are interesting since they can be used to derive upper bounds on the complexity of lower envelopes of line segments, lower envelopes of more general functions, as well as obtaining an alternate proof of the zone theorem. They are very often used to obtain simpler proofs of the complexity of various combinatorial structures as well. In short, they are very useful! For the origin of this notion see [15] and [16].

We first give some results for $\lambda_s(n)$ for $s = 1, 2, 3$.

Lemma 16 $\lambda_1(n) = n$

PROOF: We note that $(1, 2, \ldots, n)$ is a $DS(1, n)$-sequence so $\lambda_1(n) \geq n$. To prove the reverse inequality let U be a $DS(1, n)$-sequence. U cannot take the form $(\ldots a \ldots b \ldots a \ldots)$ for any $a \neq b$ and therefore the elements in U must be distinct. Hence $|U| \leq n$. □

Lemma 17 $\lambda_2(n) = 2n - 1$

PROOF: We note that $(1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1)$ is a $DS(2, n)$-sequence so $\lambda_2(n) \geq 2n - 1$. The reverse inequality is proved by induction. The case $n = 1$ is obvious. Assume the statement is true for $n - 1$ and let U be a $DS(2, n)$-sequence (over the alphabet consisting of $1, 2, \ldots n$). VLOG[3] we can assume that the leftmost occurrence of i is to the left of j iff[4] $i < j$. This implies that there can be only one occurrence of n in U otherwise there must be a subsequence of the form $(i, \ldots, n, \ldots, i, \ldots, n)$ which is forbidden. Remove the single occurrence of n, and if the symbols adjacent to n are the same remove also one of these from U. The resulting sequence is a $DS(2, n - 1)$-sequence and it is one or two elements shorter than U. The induction hypothesis yields $|U| \leq 2(n - 1) - 1 + 2 = 2n - 1$. □

Lemma 18 $\lambda_3(n) \leq 2n(1 + \log n)$

[3] VLOG="without loss of generality"
[4] iff="if and only if"

PROOF: We prove this by induction. Let a be the least frequently occurring character in the sequence. An upper bound on the number of times it occurs is $\frac{\lambda_3(n)}{n}$. Deleting all copies of a yields a sequence that can potentially have consecutive characters. However, at most 2 such consecutive characters are generated (one each, from the first and last occurrence of a). Otherwise, the sequence has the form $\ldots a \ldots bab \ldots a \ldots$ and is not a DS sequence or order 3. The remaining sequence is a $DS(n-1,3)$ sequence of length $\lambda_3(n-1)$. Hence, an upper bound on $\lambda_3(n)$ is $\lambda_3(n-1) + \frac{\lambda_3(n)}{n} + 2$. Simplifying, we get $(1-1/n)\lambda_3(n) \leq \lambda_3(n-1) + 2$.

This is the same as $\frac{\lambda_3(n)}{n} \leq \frac{\lambda_3(n-1)}{n-1} + \frac{2}{n-1}$. Expanding the recurrence gives the upper bound. □

Zone Theorem

We now study some applications of these sequences. The first application is to obtain a new proof of the zone theorem.

Let ℓ be the new line that is being inserted. We would like to prove that the total complexity of the faces that ℓ intersects is $O(n)$. We do this as follows. Assume ℓ is horizontal. We will show that the complexity of the portion of the faces "above" ℓ is $4n$. Assume no two lines are parallel. With this assumption the highest vertex on any face is unique (since no other line is horizontal). As we traverse the faces in clockwise order, we output the line number each edge belongs to. Lines are labelled with distinct numbers. We create two sequences, a left edge sequence and a right edge sequence. For the edges before the highest point in the clockwise traversal of a face, the line numbers are appended to the left sequence (LS), and the edges after the highest point in the traversal, the line numbers are appended to the right sequence (RS).

We claim that each list is $DS(n,2)$ sequence. The first condition is easy to establish. The second condition is established by a "picture-proof". Consider the right sequence. If it contains $a \ldots b \ldots a$ then it cannot contain b again. Make a picture!

Figure

This proves that the total complexity of the faces above ℓ is at most $4n$. This gives us a bound of about $8n$. Since the lines that cross ℓ were counted twice, we can reduce this to $6n$. However, the tight upper bound is $\frac{11}{2}n$.

Closest Pair/Nearest Neighbors of Moving Points

Let P be set of n points in the plane that are moving along predefined trajectories. Let $x_i(t)$ and $y_i(t)$ denote the position of point p_i at time t. Assume that these functions are polynomials of a fixed degree s. We fix our attention on point p_i, and let $g_{ij}(t)$ be the square of the distance from p_i to p_j at time t. Clearly, g_{ij} is a polynomial of degree $2s$. Let $G_i(t) = \min_{j \neq i} g_{ij}(t)$. The function $G_i(t)$ is a minima of $n-1$ functions. Each pair of functions g_{ij} and g_{ik} pairwise intersect in at most $2s$ points, since that is the degree of the polynomial defining them. We now claim that the number of intersection points on the lower envelope is at most $\lambda_{2s}(n-1)$. This is because if two functions appear $2s+2$ times alternately on the lower envelope is at most $\lambda_{2s}(n-1)$. This is because if two functions appear $2s+2$ times alternately on the lower envelope, then they must have $2s+1$ intersection points. This is an upper bound on the number of times the nearest neighbor of p_i can change.

6 Appendix: Dahlberg's Four Vertex Theorem (discrete version)

6.1 Curvature and Circle of Curvature

In this section we shall discuss an analogue of the four-vertex theorem. To state the theorem we need some notation.

Let γ be a Jordan curve of class C^2 in $\mathbf{R}^2$ with positive orientation and let Ω denote the interior of γ. For $p \in \gamma$ set

$$\Gamma(p) = \text{t}he\ largest\ circle\ C\ \text{w}ith\ p \in C\ \text{a}nd\ the\ interior\ of\ C\ \text{i}nside\ \Omega$$

$$n(p) = \text{t}he\ unit\ inward\ normal\ of\ \gamma\ \text{a}t\ p$$

and

$$k(p) = the\ \ \mathbf{c}urvature\ \ of\ \ \gamma\ \ \text{at}\ \ p.$$

Figure

If $k(p) \neq 0$ then

$$R(p) = \frac{1}{k(p)}$$

denotes the **radius of curvature**. For $k(p) \neq 0$ we set

$$\rho(p) = \{w \in \mathbf{R}^2 : |w - z(p)| = |R(p)|\},$$

where $z(p) = p + R(p)n(p)$ and for $k(p) = 0$ we set

$$\rho(p) = \{w \in \mathbf{R}^2 :< w - p, n(p) >= 0\}.$$

If $k(p) > 0$ we set

$$\omega(p) = \{w \in \mathbf{R}^2 : |w - z(p)| \leq R(p)\},$$

if $k(p) < 0$

$$\omega(p) = \{w \in \mathbf{R}^2 : |w - z(p)| \geq |R(p)|\}$$

and if $k(p) = 0$

$$\omega(p) = \{w \in \mathbf{R}^2 :< w - p, n(p) > \geq 0\}.$$

The set $\rho(p)$ is called the **circle of curvature** for γ at p. Finally for two points $p, q \in \mathbf{R}^2$ we let $[p, q]$ denote the line segment joining p and q and $(p, q) = [p, q] \setminus \{p, q\}$.

The following theorem is due to Guggenheimer [26] but the ideas for the proof comes mainly from [52].

Theorem 29 *Let γ be a Jordan curve of class C^2 in $\mathbf{R}^2$. Assume γ is not a circle. Then there are two points $p_0, p_1 \in \gamma$, $p_0 \neq p_1$, such that $k(p_0) > 0$, $k(p_1) > 0$, $\Gamma(p_0) = \rho(p_0)$ and $\Gamma(p_1) = \rho(p_1)$. Also, there are two points $q_0, q_1 \in \gamma$, $q_0 \neq q_1$, such that both $\omega(q_0)$ and $\omega(q_1)$ contain Ω. Furthermore the circles $\Gamma(p_0), \Gamma(p_1), \rho(q_0)$ and $\rho(q_1)$ are all pairwise distinct.*

Figure

The proof of the theorem is a consequence of the lemmas below.

Lemma 19 *Let $a, b \in \mathbf{R}^2$ and let $A = \{w \in \mathbf{R}^2 : |w - a| = r_a\}$, $B = \{w \in \mathbf{R}^2 : |w - b| = r_b\}$ be two circles with $r_a > 0, r_b > 0$. Assume $p_a \in A$, $p_b \in B$ satisfy $|p_a - b| > r_b$ and $|p_b - a| > r_a$. Then $[a, p_a] \bigcap [b, p_b] = \emptyset$.*

Proof: Assume $w \in [a, p_a] \bigcap [b, p_b]$ and assume the ordering is such that $|w - p_b| \leq |w - p_a|$. Then

$$|a - p_b| \leq |a - w| + |w - p_b| \leq |a - w| + |w - p_a| = r_a.$$

Figure

This contradicts the assumption $|a - p_b| > r_a$, which establishes the lemma. □

Lemma 20 *Let γ be a Jordan curve of class C^2. Assume $p \in \gamma$ and $k(p) > 0$. Let r be the radius of $\Gamma(p)$. Then $r \leq R(p)$.*

PROOF: Set $R = R(p)$ and $k = k(p)$. We may without loss of generality assume $p = 0$ and that the unit tangent to γ at p equals $(1, 0)$. We can find C^2-functions f and g whose graphs represent γ and $\Gamma(0)$ near 0. Clearly

$$f(x) = \frac{k}{2}x^2 + o(x^2)$$

and

$$g(x) = \frac{1}{2r}x^2 + o(x^2).$$

Since the interior of $\Gamma(p)$ is inside γ we must have

$$f(x) \leq g(x) \ \text{ for } x \text{ near } 0.$$

Hence

$$k \leq \frac{1}{r},$$

which yields the lemma. □

Lemma 21 *Let γ be a Jordan curve of class C^2. If $p \in \gamma$ and $q \in \Gamma(p) \bigcap \gamma$ then $\Gamma(p) = \Gamma(q)$.*

PROOF: If $q \in \Gamma(p) \bigcap \gamma$ then $\Gamma(p)$ and γ are tangent at q, which easily yields that $\Gamma(p) = \Gamma(q)$. □

Lemma 22 *Let γ be a Jordan curve of class C^2. Assume $p_k, q_k \in \gamma$, $k = 1, 2, \ldots$, have the property that $p_k \neq q_k$, $q_k \in \Gamma(p_k)$ and $p_k \to p^*$, $q_k \to p^*$ for some $p^* \in \gamma$ Then $k(p^*) > 0$ and $\Gamma(p^*) = \rho(p^*)$.*

PROOF: Let r_k, z_k be the radius and center of $\Gamma(p_k)$. Let $T(p)$ denote the unit tangent to γ at p. We note that $\dfrac{T(q_k)}{T(p_k)} \to 1$. We may assume that there is an $r^* > 0$ and z^* such that $r_k \to r^*$ and $z_k \to z^*$. We equip γ with the arc-length parametrization and we assume $\gamma(0) = p^*$, $\gamma(s_k) = p_k$, $\gamma(t_k) = q_k$ with $s_k < t_k$. Clearly $s_k \to 0$ and $t_k \to 0$. We may also assume $T(q_k) = e^{i\theta_k}T(p_k)$, $|\theta_k| < \frac{\pi}{4}$. Notice that $\theta_k \to 0$. Since $T(p_k)$ and $T(q_k)$ are tangent

to $\Gamma(p_k)$ we must have $\theta_k > 0$ and $|p_k - q_k| = 2r_k \sin(\frac{\theta_k}{2})$. From the fact that γ is C^2 follows that there is an $r_0 > 0$ such that $r_k \geq r_0$ for all k. Hence $r^* > 0$. From the smoothness of γ follows that

$$k(p^*) = \lim_{k\to\infty} \frac{\theta_k}{|p_k - q_k|} = \frac{1}{r^*} > 0.$$

Therefore $R(p^*) = r^*$. Put

$$\Sigma = \{w \in \mathbf{R}^2 : |w - z^*| = r^*\}.$$

Then $p^* \in \Sigma$ and the interior of Σ is contained inside γ. Hence Σ and γ are tangent at p^*. Therefore Σ and $\rho(p^*)$ are tangent at p^* and since they have the same radius they must coincide. Consequently the interior of $\rho(p^*)$ is inside γ, which means that $R(p^*) \leq \delta$, where δ is the radius of $\Gamma(p^*)$. Since $\delta \leq R(p^*)$ by Lemma 20 it follows that $\rho(p^*) = \Gamma(p^*)$. □

Lemma 23 *Let γ be a Jordan curve of class C^2. Let $p \in \gamma$. If $\Gamma(p) \bigcap \gamma$ is connected then $k(p) > 0$ and $\Gamma(p) = \rho(p)$.*

PROOF: Set $E = \Gamma(p) \bigcap \gamma$. If E contains two distinct points p and q then E is a closed interval with non-empty interior. Hence γ coincides with a circular arc on E which yields the result in this case. We are now left with the case $E = \{p\}$.

Figure

Let δ denote the radius of $\Gamma(p)$ and let Ω be the interior of γ. For $r > 0$ set $z_r = p + rn(p)$, $A(r) = \{w \in \mathbf{R}^2 : |w - z_r| = r\}$ and $\Delta(r) = \{w \in \mathbf{R}^2 : |w - z_r| < r\}$. Then $p \in A(r)$ and $A(\delta) = \Gamma(p)$. Set $k = k(p)$ and assume

$$k < \frac{1}{\delta}. \tag{11}$$

We may without loss of generality assume $p = 0$ and that the unit tangent to γ at p equals $(1, 0)$. We can now represent γ and $A(r)$ near 0 as the graphs of C^2-functions f and g with

$$f(x) = \frac{k}{2}x^2 + o(x^2)$$

and

$$g(x) = \frac{1}{2r}x^2 + o(x^2).$$

It follows from (11) that there is an open disk D centered at 0 and a neighborhood I of δ such that $\Delta(r) \bigcap \overline{D}$ is contained in Ω for all $r \in I$. Since $\overline{\Delta(\delta)} \bigcap (\mathbf{R}^2 \setminus D)$ has a positive distance to γ there is a neigborhood J of δ such that $\overline{\Delta(r)} \bigcap (\mathbf{R}^2 \setminus D)$ is contained in Ω for all $r \in J$. We now conclude that if (11) holds then there is an $r > \delta$ such that $\Delta(r) \subset \Omega$. This contradicts the definition of $\Gamma(p)$. We must therefore have $k \geq \frac{1}{\delta}$. In particular, $k > 0$ and from Lemma 20 follows $k = \frac{1}{\delta}$, which establishes the lemma. □

Lemma 24 *Let γ be a Jordan curve of class C^2. Assume $p \in \gamma$ and $\gamma \bigcap \Gamma(p) \neq \gamma$. Let I be a component of $\gamma \setminus (\gamma \bigcap \Gamma(p))$. If $q \in I$ then $\Gamma(q) \bigcap \gamma \subset I$.*

Figure

PROOF: Since $q \in \Gamma(p)$ it follows from Lemma 21 that $\Gamma(p) \bigcap \Gamma(q) \bigcap \gamma = \emptyset$. Let a, b denote the end points of I and assume there is a $w \in \Gamma(q) \bigcap \gamma$ such that $w \notin I$. Since $a, b \in \Gamma(p)$ we have $\{a, b\} \bigcap \Gamma(q) = \emptyset$. Let p^*, q^* be the centers of $\Gamma(p)$ and $\Gamma(q)$. Let γ_1, γ_2 be the polygons with vertices $\{b, p^*, a\}$ and $\{q, q^*, w\}$. Let γ^* be the Jordan curve one gets by following I from a to b and joining back to a by following γ_1.

Figure

Since w is outside γ^* it follows that γ_1 and γ_2 intersect. Since $\{a, b\} \bigcap \Gamma(q) =$

$\emptyset$ and $\{q, w\} \bigcap \Gamma(p) = \emptyset$ it follows from Lemma 19 that this is impossible. The lemma is proved. $\square$

Lemma 25 *Let γ be a Jordan curve of class C^2. Assume γ is not a circle. Then there are two points p_0, p_1 on γ such that $k(p_0) > 0, k(p_1) > 0, \Gamma(p_0) = \rho(p_0)$ and $\Gamma(p_1) = \rho(p_1)$. Furthermore, $\Gamma(p_0)$ and $\Gamma(p_1)$ do not coincide.*

PROOF: Let $N(p)$ be the number of components of $\gamma \setminus (\Gamma(p) \bigcap \gamma)$, $p \in \gamma$. Then $1 \leq N(p) \leq \infty$, since γ is not a circle. If $N(p) = 1$ for all $p \in \gamma$ then any pair of distinct $p_0, p_1 \in \gamma$ will work by Lemma 23. If $N(p) > 1$ for some $p \in \gamma$ let I be any component of $\gamma \setminus (\Gamma(p) \bigcap \gamma)$. We claim that there is a $p^* \in I$ such that $k(p^*) > 0$ and $\Gamma(p^*) = \rho(p^*)$. Let q be the midpoint of I. If $N(q) = 1$ we are done by Lemma 23. If $N(q) > 1$ then $\gamma \bigcap \Gamma(q) \subset I$ by Lemma 24. Let I^* be any component of $\gamma \setminus (\Gamma(q) \bigcap \gamma)$ that is contained in I. Then

$$\sigma(I^*) \leq \frac{1}{2}\sigma(I),$$

where σ denotes the arc-length measure. Since $\Gamma(p) \bigcap \Gamma(q) \bigcap \gamma = \emptyset$ we have that $\overline{I^*} \subset I$. Recall that I is an open set in γ. Repeating this process gives us a sequence of nested intervals I_k such that

$$I_{k+1} \subset I_k \subset I^*$$

and

$$\sigma(I_{k+1}) \leq \frac{1}{2}\sigma(I_k).$$

If a_k, b_k denote the end points of I_k then $b_k \in \Gamma(a_k)$, $a_k \neq b_k$. Clearly we can find a $p^* \in I^*$ such that $a_k \to p^*$ and $b_k \to p^*$. Hence the claim follows from Lemma 22 $\square$

PROOF: (Proof of Theorem 29) The existence of p_0, p_1 follows from Lemma 25. Pick a point z_0 inside γ. Set $F(z) = \dfrac{1}{z - z_0}$ and let $\gamma^* = F(\gamma)$. Notice that F is a Möbius mapping that maps the outside of γ onto the inside of γ^*. Furthermore, F preserves circles of curvature. Using Lemma 25 for the curve γ^* completes the proof. $\square$

6.2 Circles of Curvature for Convex Polygons

We shall now give the discrete version of the four-vertex theorem.

Let γ be a closed simple polygon. Let $V(\gamma)$ denote the set of its vertices $\{P_0, P_1, \ldots, P_{N-1}\}$. For $P \in V(\gamma)$ let $\rho(P)$ be the unique circle passing through P and its two immediate neighboring vertices. $\rho(P)$ is called the **discrete circle of curvature** at the vertex P. Let $\omega(P)$ be the closed disk formed by $\rho(P)$ together with its interior. We have the following analogue of Theorem 29 for strictly convex polygons.

Theorem 30 (Discrete Four-Vertex Theorem (preliminary form)) *Let γ be a strictly convex polygon in $\mathbf{R}^2$. Assume that $V(\gamma)$ does not lie on a circle. Then there are $P, Q \in V(\gamma)$ such that*

$$\mathrm{Int}(\omega(P)) \bigcap V(\gamma) = \mathrm{Int}(\omega(Q)) \bigcap V(\gamma) = \emptyset.$$

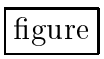
Also there are $P^*, Q^* \in V(\gamma)$ such that

$$V(\gamma) \subset \omega(P^*)$$

and

$$V(\gamma) \subset \omega(Q^*).$$

Furthermore the circles $\rho(P), \rho(P^*), \rho(Q)$ and $\rho(Q^*)$ are all pairwise distinct.



Recall that a convex polygon is **strictly convex** if no three vertices lie on a line.

From now on we let γ be a strictly convex polygon with N vertices. For $P \in V(\gamma)$ let $\mathcal{F}(P)$ denote the family of all closed disks U such that $P \in \partial U$, $V(\gamma) \subset U$ and ∂U contains at least three points in $V(\gamma)$. Similarly, let $\mathcal{G}(P)$ be the family of all open disks U such that $P \in \partial U$, $V(\gamma) \bigcap U = \emptyset$ and ∂U contains at least three points in $V(\gamma)$.

Lemma 26 *For all $P \in V(\gamma)$ we have that both $\mathcal{F}(P)$ and $\mathcal{G}(P)$ are non-empty.*

PROOF: If p and q are two distinct points in $\mathbf{R}^2$ we let $A(p,q)$ denote the class of closed disks U such that $\{p,q\} \subset \partial U$. Notice that if L denotes the line through p and q then

$$\mathbf{R}^2 \setminus (L \setminus [p,q]) = \bigcup_{U \in A(p,q)} U. \tag{12}$$

figure

Assume $P \in V(\gamma)$. Since γ is strictly convex there is a closed disk W such that $V(\gamma) \subset W$ and $P \in \partial W$. Pick $Q_1 \in \partial W$ with $Q_1 \neq P$. From (12) it follows that there is a $W_1 \in A(P,Q_1)$ such that $V(\gamma) \subset W_1$ and $(\partial W_1 \setminus \{P\}) \bigcap V(\gamma) \neq \emptyset$. Let $Q_2 \neq P$ belong to $\partial W_1 \bigcap V(\gamma)$. Again using (12) we see that there is a $W_2 \in A(P_1,Q_2)$ such that $V(\gamma) \subset W_2$ and $(\partial W_2 \setminus \{P,Q_2\}) \bigcap V(\gamma) \neq \emptyset$. Hence $\mathcal{F}(P) \neq \emptyset$.

A similar argument shows that $\mathcal{G}(P) \neq \emptyset$. □

Remark: The proof that $\mathcal{G}(P) \neq \emptyset$ does not require that γ is strictly convex.

A subset E of $V(\gamma)$ is said to be **connected** if E consists of a sequence of consecutive vertices. In this case E is called an **interval**. If $E \subset V(\gamma)$ is an interval of the form $E = \{P_j, P_{j+1}, \ldots, P_k\}$ we let $\hat{E} = E \bigcup \{P_{j-1}, P_{k+1}\}$. Here the vertices are counted modulo N. Clearly $\hat{E}$ is also an interval.

Lemma 27 *Let $P \in V(\gamma)$ and let $U \in \mathcal{F}(P)$. Assume $I \neq \emptyset$ is a maximal interval contained in $V(\gamma) \setminus \partial U$. If $Q \in I$ and $W \in \mathcal{F}(Q)$ then*

$$\partial W \bigcap V(\gamma) \subset \hat{I}.$$

PROOF: There is nothing to prove if $\hat{I} = V(\gamma)$. Assume therefore that $\hat{I}$ is a proper subset of $V(\gamma)$. Let $a, b \in V(\gamma)$ be such that $a \neq b$ and $\hat{I} = I \bigcup \{a, b\}$. Let L denote the line through a and b. Then L divides $\mathbf{R}^2$ into two open half planes H and H^*. Assume that $Q \in H$. Then $I = H \bigcap V(\gamma)$ by convexity. Notice that $\{a, b\} \subset W$. Hence $\partial W \bigcap U \subset \overline{H}$.

figure

In particular,

$$\partial W \bigcap V(\gamma) \subset (\partial W \bigcap U) \bigcap V(\gamma) \subset \hat{I}.$$

□

Lemma 28 *Assume $V(\gamma)$ is not contained in a circle. Then there are $P^*, Q^* \in V(\gamma)$ such that*

$$V(\gamma) \subset \omega(P^*),$$

$$V(\gamma) \subset \omega(Q^*)$$

and

$$\rho(P^*) \neq \rho(Q^*).$$

PROOF: Assume that there exists a $P \in V(\gamma)$ and $U \in \mathcal{F}(P)$ such that $V(\gamma) \bigcap \partial U$ is not an interval. Let $I_1, \ldots, I_n$, $n \geq 2$, be the maximal non-empty intervals in $V(\gamma) \setminus U$. By repeated use of Lemma 27 follows that there are points $Q_k \in I_k$, $1 \leq k \leq n$, such that if $W_k \in \mathcal{F}(Q_k)$ then $V(\gamma) \bigcap W_k \subset \hat{I_k}$ and $V(\gamma) \bigcap W_k$ is connected. Also, $Q_j \notin W_k$ if $j \neq k$. Since $W_k \bigcap V(\gamma)$ is connected there is a $Q_k^* \in W_k \bigcap V(\gamma)$ such that $\omega(Q_k^*) = W_k$. If the assumption does not hold then the result is trivial. □

Lemma 29 *Let $P \in V(\gamma)$ and let $U \in \mathcal{G}(P)$. Assume $I \neq \emptyset$ is a maximal interval contained in $V(\gamma) \setminus \partial U$. If $Q \in I$ and $W \in \mathcal{G}(Q)$ then*

$$\partial W \bigcap V(\gamma) \subset \hat{I}.$$

PROOF: There is nothing to prove if $\hat{I} = V(\gamma)$. Assume $\hat{I} \neq V(\gamma)$. Suppose there is a $Q^* \in (\partial W \bigcap V(\gamma)) \setminus \hat{I}$. Let a and b be the end points of I. Then $a, b \in \partial U \bigcap V(\gamma)$. Hence $W \bigcap \{a, b\} = \emptyset$. Since $Q \in \partial W$ but $Q \notin \partial U$ neither U nor W can be contained in the other. Let L be the line through a and b. Then L separates $\mathbf{R}^2$ into two half planes H and H^*, say $Q \in H$. Then $Q^* \in H^*$ since $\hat{I} = V(\gamma) \bigcap \overline{H}$. Since $[Q, Q^*] \bigcap L \subset (a, b)$ by the strict convexity of γ we have that $\overline{W} \bigcap L \subset [a, b]$.

figure

Hence $\partial W \bigcap \partial U \subset \overline{H}$ so $\overline{W} \bigcap H^* \subset U$. Since U is open and $U \bigcap V(\gamma) = \emptyset$ the lemma follows by contradiction. □

Lemma 30 *Assume $V(\gamma)$ is not contained on a circle. Then there are P^*, Q^* in $V(\gamma)$ such that*

$$\text{Int}(\omega(P^*)) \bigcap V(\gamma) = \text{Int}(\omega(Q^*)) \bigcap V(\gamma) = \emptyset$$

and

$$\rho(P^*) \neq \rho(Q^*).$$

PROOF: A straight-forward modification of the proof of Lemma 28 gives the result. □

PROOF: (Proof of Theorem 30) The theorem is a direct consequence of Lemma 28 and Lemma 30. □

Corollary 3 *Let R be a strictly convex quadrilateral with vertex set S. Assume that S is not contained in a circle. Then there are two diametrically opposite vertices P, P^* such that*

$$S \subset \omega(P)$$

and

$$S \subset \omega(P^*).$$

Furthermore, the discrete circles of curvature at the other pair of diametrically opposite vertices contain no point from S in there interiors.

PROOF: From the Discrete Four-Vertex Theorem follows that there is a $P \in S$ such that $S \subset \omega(P)$. Let P^* be the vertex diametrically opposite to P. From Lemma 27 and the fact that $S \setminus \rho(P) = \{P^*\}$ follows that $S \subset \omega(P^*)$. The remaining part of the corollary follows from the Discrete Four-Vertex Theorem. □

6.3 Cauchy's Theorem on Convex Polygons

This section will be devoted to Cauchy's result on convex polygons that he developed for his proof of the rigidity of convex polyhedrons. We only consider polygons in $\mathbf{R}^2$.

Lemma 31 *Let $\gamma = \{P_0, \ldots, P_{N+1}\}$ and $\Gamma = \{Q_0, \ldots, Q_{N+1}\}$ be two convex polygons. Let α_k, β_k denote the interior angles of γ, Γ at the k:th vertex. Assume that $\alpha_k \leq \beta_k$ for $1 \leq k \leq N$ and $|P_{k+1} - P_k| \leq |Q_{k+1} - Q_k|$ for $0 \leq k \leq N$. Then $|P_{N+1} - P_0| \leq |Q_{N+1} - Q_0|$ with equality if and only if $\alpha_k = \beta_k$ for all $k \in \{1, \ldots, N\}$.*

PROOF: (See [54]) Let P_m be the vertex of γ that is farthest away from the line through the side $[P_0, P_{N+1}]$.

figure

Select a coordinate system (x, y) such that P_0, P_{N+1} are on the x-axis and P_m falls on the positive y-axis. Let x_k denote the abscissa of P_k and assume $x_{N+1} > x_0$. Set $s_k = |P_{k+1} - P_k|$ and $\theta_k(t) = (1-t)\alpha_k + t\beta_k$. We shall now deform γ into the polygon $\gamma_t = \{P_0(t), \ldots, P_{N+1}(t)\}$, $t \in [0, 1]$. We will require that $\gamma_0 = \gamma$ and $P_m(t) = P_m$ for all $t \in [0, 1]$ We also demand that $|P_{k+1}(t) - P_k(t)| = s_k$ for $0 \leq k \leq N$ and that γ_t has the interior angle $\theta_k(t)$ at the k:th vertex for $1 \leq k \leq N$. Letting $C_m(t)$ denote the convex cone with apex P_m determined by the m:th vertex of γ_t we require that $C_m(s) \subset C_m(t)$ for $0 \leq s \leq t \leq 1$.

Choose $\phi_k(t)$ such that

$$P_{k+1} - P_k = s_k e^{i\phi_k(t)}.$$

We can select the ϕ_k:s such that $0 \leq \phi_k \leq \pi$ for $0 \leq k < m$ and $-\pi \leq \phi_k \leq 0$ for $m \leq k \leq N$. Notice that $\phi_m, \ldots, \phi_N$ are non-decreasing and $\phi_0, \ldots, \phi_{m-1}$ are non-increasing functions of t in $[0,1]$. Denote by $x_k(t)$ the abscissa of $P_k(t)$. Since

$$x_{N+1}(t) = \Sigma_{k=m}^{N} s_k \cos \phi_k(t)$$

we have that $x_{N+1}(t)$ is non-decreasing in $[0,1]$. Similarly we have that $x_0(t)$ is non-increasing. Therefore $x_0(t) \leq x_0$ and $x_{N+1}(t) \geq x_{N+1}$ for $t \in [0,1]$. Hence

$$|P_{N+1}(t) - P_0(t)| \geq |x_{N+1}(t) - x_0(t)| = x_{N+1}(t) - x_0(t)$$

$$\geq x_{N+1} - x_0 = |P_{N+1} - P_0|.$$

The lemma is an immediate consequence of this estimate. □

Let $\mathbf{Z}_n = \{0, 1, \ldots, n-1\}$. We say that a set $I \subset \mathbf{Z}_n$ is an **interval** if there are integers a, b such that $a \leq b$ and $I = \{k \bmod n : a \leq k \leq b\}$. Let W_n be the class of sequences $\xi = \{\xi_j\}_{j=0}^{n-1}$ such that $\Sigma_{j=0}^{n-1} \xi_j = 0$ but $\xi_k \neq 0$ for some k. If $\xi \in W_n$ then $R(\xi)$ denotes the number of distinct maximal intervals $I \subset \mathbf{Z}_n$ such that $\xi_k \geq 0$ for all $k \in I$. We leave it to the reader to show that $R(\xi) = R(-\xi)$ for all $\xi \in W_n$. This means that a sequence $\xi \in W_n$ has $2R(\xi)$ sign changes.

The following result is called **Cauchy's Lemma**.

Theorem 31 (Cauchy's Lemma) *Let $\gamma = \{P_0, \ldots, P_{N-1}\}$ and $\Gamma = \{Q_0, \ldots, Q_{N-1}\}$ be two closed convex polygons. Let α_k, β_k be the interior angles of γ, Γ at the k:th vertex and set $\theta_k = \beta_k - \alpha_k$. Assume $|Q_{k+1} - Q_k| = |P_{k+1} - P_k|$ for $0 \leq k < N$. Also assume that $\beta_k \neq \alpha_k$ for some k. Set $\theta = \{\theta_k\}_{k=0}^{N-1}$. Then $\theta \in W_N$ and has at least 4 sign changes.*

Proof: We remark that the result is trivial if $N = 3$. Since γ and Γ both have N vertices we have $\Sigma_{k=0}^{N-1} \alpha_k = \Sigma_{k=0}^{N-1} \beta_k = (N-2)\pi$. So $\theta \in W_n$. Since $\theta \in W_n$ we have that $R(\theta) \geq 1$. It is enough to show that $R(\theta) \geq 2$. Assume that $R(\theta) = 1$. Then we can find $a, b \in \mathbf{Z}_N$ and disjoint intervals $I_1, I_2 \subset \mathbf{Z}_N$ such that $I_1 \bigcup I_2 \bigcup \{a, b\} = \mathbf{Z}_N$ with $\theta_k \geq 0$ for all $k \in I_1$ and $\theta_k \leq 0$ for all $k \in I_2$, Applying Lemma 31 to I_1 we have that $|Q_b - Q_a| \geq |P_b - P_a|$. Moreover using Lemma 31 on I_2 shows that $|Q_b - Q_a| \leq |P_b - P_a|$. Since now

$|Q_b - Q_a| = |P_b - P_a|$ we must have that $\theta_k = 0$ for $k \in I_1 \bigcup I_2$. Hence $\theta_k \neq 0$ if and only if $k \in \{a, b\}$. Select now $a^*, b^* \in \mathbf{Z}_N$ such that $\{a^*, b^*\} \bigcap \{a, b\} = \emptyset$ and disjoint intervals J_1, J_2 such that $J_1 \bigcup J_2 \bigcup \{a^*, b^*\} = \mathbf{Z}_N$. Repeating the above argument shows that $\theta_k = 0$ for $k \in \{a, b\}$, which contradicts the fact that $\theta \in W_N$. Hence $R(\theta) \geq 2$, which yields the theorem. $\square$

We shall now consider the smooth version of Cauchy's Lemma.

Let $I = [a, b]$ be a closed bounded interval. Let $\mathcal{F}(I)$ denote the class of one–to–one C^1-mappings $\gamma : I \to \mathbf{R}^2$ with $|\frac{d\gamma}{dt}| = 1$ and $\frac{d\gamma}{dt}$ Lipschitz continuous such that γ together with the line segment $[\gamma(a), \gamma(b)]$ bounds a convex domain.

figure

We also require that the curvature of γ is non-negative *a.e.*

Lemma 32 *Let $I = [a, b]$ and $\gamma, \Gamma \in \mathcal{F}(I)$. Let γ, Γ have the curvature k and K. Assume $k \geq K$ a.e. in I. Then $|\gamma(b) - \gamma(a)| \leq |\Gamma(b) - \Gamma(a)|$ with equality if and only if $k = K$ a.e. in I.*

Proof: We set $\dot{\gamma} = \frac{d\gamma}{dt}$ and $\dot{\Gamma} = \frac{d\Gamma}{dt}$. We begin by picking a $\rho \in (a, b)$ such that

$$\dot{\gamma}(\rho) = \frac{\gamma(b) - \gamma(a)}{|\gamma(b) - \gamma(a)|}.$$

We may without loss of generality assume that $\gamma(\rho) = \Gamma(\rho) = 0$ and $\dot{\gamma}(\rho) = \dot{\Gamma}(\rho) = (1, 0)$. Define α, β by the requirements that $\alpha(\rho) = \beta(\rho) = 0$, $\frac{d\alpha}{dt}(\rho) = k(\rho)$ and $\frac{d\beta}{dt}(\rho) = K(\rho)$. Then $\gamma(s) = \int_\rho^s e^{i\alpha(t)}\, dt$ and $\Gamma(s) = \int_\rho^s e^{i\beta(t)}\, dt$. Let $\gamma(t)$ and $\Gamma(t)$ have the abscissas $x(t)$ and $\xi(t)$.

figure

By convexity $0 \leq \alpha(t) \leq \pi$ for $t \in [\rho, b]$. Hence $0 \leq \beta(t) \leq \alpha(t) \leq \pi$ for $t \in [\rho, b]$. If $\rho \leq s \leq b$ then

$$\xi(s) = \int_\rho^s \cos\beta(t)\, dt \geq \int_\rho^s \cos\alpha(t)\, dt = x(s).$$

In particular, $\xi(b) \geq x(b)$. Similarly $\xi(a) \leq x(a)$. Hence

$$|\Gamma(b) - \Gamma(a)| \geq |\xi(b) - \xi(a)| = \xi(b) - \xi(a)$$

$$\geq x(b) - x(a) = |\gamma(b) - \gamma(a)|,$$

which yields the lemma. □

We shall now make a more precise definition of the class of smooth convex closed curves of length L. For $L > 0$ we let $\mathcal{F}_L$ be the class of C^2-mappings $\gamma : \mathbf{R} \to \mathbf{R}^2$ such that $|\frac{d\gamma}{dt}| = 1$, $\gamma(t+L) = \gamma(t)$ for all $t \in \mathbf{R}$, γ is one–to–one on $[0, L)$ and γ has non-negative curvature.

The following version of Cauchy's Lemma is due to Blaschke [5] but our proof follows Bol [6].

Theorem 32 *Let $\gamma, \Gamma \in \mathcal{F}_L$ have the curvature k and K. Assume $k(s) \neq K(s)$ for some s. Set $f = k - K$. Then there are four points s_i, $s_1 < s_2 < s_3 < s_4$, $s_4 - s_1 < L$ such that $f(s_1) > 0$, $f(s_2) < 0$, $f(s_3) > 0$ and $f(s_4) < 0$.*

Remark: The theorem says that f has 4 distinct zeros, i.e. γ and Γ have equal curvatures in at least 4 points. Moreover f has at least two local maxima and two local minima.

Proof: Assume the conclusion does not hold. Then we can find $a, b \in \mathbf{R}$ with $a < b < a + L$ such that $f \geq 0$ in $I_1 = [a, b]$ and $f \leq 0$ in $I_2 = [b, a + L]$. Using Lemma 32 on I_1 we see that $|\gamma(b) - \gamma(a)| \leq |\Gamma(b) - \Gamma(a)|$. Using Lemma 32 on I_2 gives $|\gamma(b) - \gamma(a)| \geq |\Gamma(b) - \Gamma(a)|$. Hence $|\gamma(b) - \gamma(a)| = |\Gamma(b) - \Gamma(a)|$ so again by Lemma 32 we have $k = K$ *a.e.* Since k and K are continuous we must have $k(s) = K(s)$ for all $s \in \mathbf{R}$. This contradiction establishes the theorem. □

We remark that a local extreme value for the curvature of a curve is called a **vertex**.

We shall now prove the classical **four-vertex theorem**. The method of proof appears to be new.

Theorem 33 *Let $\gamma : S^1 \to \mathbf{R}^2$ be of class C^2 with $|\frac{d\gamma}{dt}| > 0$. Assume γ is simple and that γ is not a circle. Let k denote the curvature of γ. Then there are four consecutive points $p_i \in S^1$ such that*

$$\max\{k(p_1), k(p_3)\} < \min\{k(p_2), k(p_4)\}.$$

PROOF: We will assume that γ has been given the positive orientation so that $\int_{S^1} k\, d\sigma = 2\pi$, where σ denotes the arc length measure of γ. We also remark that if $k \geq 0$ then the result follows from Theorem 32. Put $m = \min k$ and $M = \max k$. We will now assume $m < 0$ since we have established the result otherwise. Pick $q_0, q_1 \in S^1$ such that $m = k(q_0)$ and $M = k(q_1)$. Notice that $M > 0$. Now q_0, q_1 split S^1 into two closed intervals I_0 and I_1.

Assume now that the conclusion of the theorem fails. Then $k|I_0$ and $k|I_1$ are both monotone. Let C be the circumcircle to γ and let R denote its radius. Then there are two distinct points $w_0, w_1 \in S^1$ such that $\gamma(w_0)$ and $\gamma(w_1)$ belong to different components of $\gamma \bigcap C$.

figure

Clearly $k(w_0) \geq \frac{1}{R}$ and $k(w_1) \geq \frac{1}{R}$. Since $k(q_0) < 0$ it follows that $q_0 \notin \{w_0, w_1\}$. Now w_0, w_1 split S^1 into two closed intervals J_0, J_1. We may choose the ordering so that $q_0 \in J_1$ but $q_0 \notin J_0$.

We now claim that

$$\min_{J_0} k \geq m^*, \tag{13}$$

where $m^* = \min\{k(w_0), k(w_1)\}$. Notice that $m^* \geq \frac{1}{R} > 0$. We shall now prove (13). If $q_1 \in J_1$ then k is monotone on J_0, which yields (13). If $q_1 \notin J_1$ then q_1 splits J_0 into two closed intervals $J_0^{'}, J_0^{''}$ with common end points at q_1. Since $q_0 \in J_1$ we have that k is monotone on both $J_0^{'}$ and $J_0^{''}$. Since $k(q_1) \geq m^*$ we see that (13) follows in this case also.

Since $\gamma(w_0)$ and $\gamma(w_1)$ belong to different components of $\gamma \bigcap C$ we must have that there is some $w \in J_0$ such that $\gamma(w) \notin C$. We may assume that $U = \{w \in J_0 : \gamma(w) \notin C\}$ is connected since otherwise we replace J_0 with the closure of a component of U.

Let Γ be the Jordan curve of class C^1 one gets by letting $\Gamma|J_0 = \gamma|J_0$ and letting $\Gamma|J_1$ be the corresponding arc on C. Then Γ is a convex Jordan curve with bounded curvature. Let L be the length of Γ and let K denote the curvature of Γ. If σ denotes the arc length measure of Γ then $\int K \, d\sigma = 2\pi$. Since $K \geq \dfrac{1}{R}$ we therefore have $L \leq 2\pi R$. However if $L = 2\pi R$ then Γ must be the circle C which is impossible by the construction of J_0.

Put $A = \dfrac{2\pi}{L}$ so A is the curvature of a circle C_A of length L. Setting $E = \{w \in J_0 : k(w) \geq A\}$ we have from the monotonicity properties of k that E is an interval. Let F be the closure of the complement of E in S^1. Then F is an interval also. We let a, b denote the common end points of E and F. Let l be the length of $\Gamma|J_0$. Let a^*, b^* be points on C_A such that the length of one of the arcs on C_A connecting a^* and b^* has length l. Applying Lemma 32 to E shows that

$$|\Gamma(b) - \Gamma(a)| \leq |a^* - b^*|.$$

Applying Lemma 32 to F gives

$$|\Gamma(b) - \Gamma(a)| \geq |a^* - b^*|.$$

Hence

$$|\Gamma(b) - \Gamma(a)| = |a^* - b^*|$$

so one more application of Lemma 32 shows that $K = A$ *a.e.* in S^1. This is impossible which establishes the theorem. □

6.4 Discrete Four Vertex Theorem

Let γ be a simple and closed polygon in $\mathbf{R}^2$. Let $V(\gamma)$ denote the vertex set of γ and let $\Omega(\gamma)$ denote the closure of the interior of γ. We shall in this section study the discrete curvature k of γ. Here k is defined as follows. If $P \in V(\gamma)$ is on the line segment between the two immediate neighbors of P we set $k(P) = 0$. Otherwise we let $\rho(P)$ be the circle that passes through P and its immediate two neighbors, i.e., $\rho(P)$ is the discrete circle of curvature at P. Let $\alpha(P)$ denote the interior angle at P and let $R(P) > 0$ be the

radius of $\rho(P)$. If $\alpha(P) < \pi$, we set $k(P) = \frac{1}{R(P)}$ and if $\alpha(P) > \pi$, we set $k(P) = -\frac{1}{R(P)}$. It is easy to give examples of polygons for which k has exactly one local maximum, so we have to find some additional conditions on a polygon for the four vertex theorem to hold for k.

We shall first make the notion of a local extreme point precise. Let $f : V(\gamma) \to \mathbf{R}$ be a function and for $P \in V(\gamma)$ let $I(f, P)$ be the largest connected set J of vertices that contains P for which $f(Q) = f(P)$ for all $Q \in J$. For $J \subset V(\gamma)$ connected we let $\hat{J}$ denote the union of J and its two immediate neighbors. We will say that P is a local maximum for f if $f(Q) < f(P)$ for all $Q \in \hat{I}(f, P) \setminus I(f, P)$. Similarly, we will say that P is a local minimum for f if $f(Q) > f(P)$ for $Q \in \hat{I}(f, P) \setminus I(f, P)$. We will say that two local extreme points P_1 and P_2 are distinct if $I(f, P_1)$ and $I(f, P_2)$ are disjoint.

For $P \in V(\gamma)$ let $\mathcal{A}(P)$ be the closed convex sector with apex at P that is determined by P and its two immediate neighbors. If $k(P) = 0$, we require that $\mathcal{A}(P)$ is that half-plane determined by P and its two immediate neighbors such that $\mathcal{A}(P) \bigcap W = \Omega(\gamma) \bigcap W$ for all sufficiently small neighborhoods of P. We will say that γ is **locally regular** if for all vertices $P \in V(\gamma)$ the center of the circle of curvature at P is contained in $\mathcal{A}(P)$. We remark that by elementary geometry one has the following sufficient conditions for local regularity.

Proposition 11 *Let γ be a simple and closed polygon. If for each vertex P the interior angle $\alpha(P)$ satisfies $\frac{\pi}{2} \leq \alpha(P) \leq \frac{3\pi}{2}$, then γ is locally regular. If all edges have the same length, then γ is locally regular.*

We can now state our discrete version of the four vertex theorem.

Theorem 34 (Discrete Four Vertex Theorem) *Let γ be a simple closed polygon. Assume that γ is locally regular and that the vertices are not on a circle. Let k denote the discrete curvature of γ. Then k has at least two local maxima and at least two local minima.*

Before proving this result, we will need some notation and some preliminary results.

Let e be an edge of the simple closed polygon γ and let l be the line that contains e. Let M be the midpoint of e. Let h_e be that closed half-plane determined by l such that for all sufficiently small neighborhoods W

of M, one has that $h_e \bigcap W = \Omega(\gamma) \bigcap W$. Let P be an endpoint of e. If $k(P) = 0$, we set $\delta(P, e) = h_e$. If $k(P) \neq 0$, we let $\omega(P)$ denote the closed disk determined by the circle of curvature at P. We set $\delta(P, e) = \omega(P) \bigcap h_e$ if $k(P) > 0$. If $k(P) < 0$, we let $\omega^*(P)$ denote the closure of the complement of $\omega(P)$ and set $\delta(P, e) = \omega^*(P) \bigcup (\omega(P) \bigcap h_e)$.

Lemma 33 *Let γ be a simple closed polygon. Assume that γ is locally regular. Let e be an edge of γ and let P, Q be the endpoints of e. Then the following holds:*

1. *If $k(P) \geq 0$, then $\delta(P, e) \subset h_e$.*
2. *If $k(P) \leq 0$, then $h_e \subset \delta(P, e)$.*
3. *If $k(P) \leq k(Q)$, then $\delta(Q, e) \subset \delta(P, e)$.*

PROOF: The first two properties are immediate consequences of the definition of $\delta(P, e)$. Hence Property 3 holds if $k(P)k(Q) \leq 0$. Assume now that $k(P)k(Q) > 0$. Let l be the line that contains e. Since γ is locally regular and $k(P)k(Q) > 0$, it follows that the circles of curvature at P and Q have their centers the same side of l. If $R(P), R(Q)$ denote the radii of the circles of curvature at P, Q, then $R(P) \geq R(Q)$ if $k(P) > 0$ and $R(P) \leq R(Q)$ if $k(P) < 0$, which yields the lemma. □

Lemma 34 *Let γ be a strictly convex polygon whose vertices do not fall on a circle. Assume that γ is locally regular. Then the discrete curvature of γ has at least two local maxima and at least two local minima.*

PROOF: We first remark that the discrete curvature k is not constant. It is enough to show that k has at least four local extreme points. Let $Q_0, Q_1 \in V(\gamma)$ be defined by $k(Q_0) = \min\{k(P) : P \in V(\gamma)\}$ and $k(Q_1) = \max\{k(P) : P \in V(\gamma)\}$. Assume that the conclusion fails. Then k must be monotone on the two maximal intervals in $V(\gamma) \setminus \{Q_0, Q_1\}$ determined by Q_0 and Q_1. For a vertex P let $\omega(P)$ be the closed disk determined by the circle of curvature at P.

Let $P \in V(\gamma)$ be different from Q_1. We can find vertices $P_1, P_2, \ldots, P_N$ such that $P_1 = Q_1, P_N = P$, $k(P_{i+1}) \leq k(P_i)$ and $P_i P_{i+1}$ is an edge for each $i \in \{1, \ldots, N-1\}$. Let e_i be the edge determined by P_i and P_{i+1} and set $h_i = h_{e_i}$, where the half-plane h_{e_i} was defined above. From the above lemma follows that $\omega(P_i) \bigcap h_i \subset \omega(P_{i+1}) \bigcap h_i$. Let Ω be the closure of the interior of γ. Since $\Omega \subset h_i$ for all $i \in \{1, \ldots, N-1\}$, we have that for all $P \in V(\gamma)$

$$\omega(Q_1) \bigcap \Omega \subset \omega(P) \bigcap \Omega.$$

Let $E = \{P \in V(\gamma) : \text{Int}(\omega(P)) \bigcap V(\gamma) = \emptyset\}$. Hence $\omega(P) = \omega(Q_1)$ for all $P \in E$.

This contradicts Theorem30, which in particular says that $\{\omega(P) : P \in E\}$ consists of at least two distinct disks. This contradiction shows that k must have at least four local extreme points, which shows the lemma. $\square$

Lemma 35 *Let ABC be a triangle and denote by $\mathcal{A}$, the closed, convex sector with vertex A determined by the triangle. Let ρ be the circumcircle of the triangle. Assume that the center of ρ belongs to $\mathcal{A}$ and assume that Δ is a closed disk containing the triangle such that $A \in \partial\Delta$. Let r, R denote the radii of Δ and ρ. Then $R \leq r$ with equality if and only if $\rho = \partial\Delta$.*

PROOF: Assume that $r < R$. By possibly shrinking Δ we may assume that $\partial\Delta$ passes through another vertex of the triangle, say $B \in \partial\Delta$. Let H be the closed half-plane determined by the line through A, B that contains C. Let δ be the closed disk with center at the midpoint of AB that goes through A. Then $\delta \bigcap H \subset \omega \bigcap H$, where ω is the closed disk determined by ρ.

Let α denote the angle at A of the triangle. If $0 < \alpha \leq \pi/2$, then the center of ρ lies inside the triangle. Therefore ρ is the smallest enclosing circle of the triangle so $R \leq r$ in this case with equality if and only if $\rho = \partial\Delta$.

We are now left with the case $\alpha > \pi/2$. We first notice that $C \notin \delta$ in this case. We claim that if z denotes the center of Δ, then $z \in H$. For if $z \notin H$, then $\Delta \bigcap H \subset \delta \bigcap H$, which is impossible since $C \in \Delta$. Hence $z \in H$ as claimed. We are now left with the situation $A, B \in \partial\Delta \bigcap \rho$ and both Δ and ρ have their centers in H. The assumption $r < R$ implies that $\rho \bigcap \Delta \bigcap \text{Int}(H) = \emptyset$, which is impossible since $C \in \Delta$. Hence $R \leq r$ with equality if and only if $\rho = \partial\Delta$. The lemma is therefore proved. $\square$

PROOF: (Proof of Theorem 34) We begin by picking $Q_0, Q_1 \in V(\gamma)$ such that $k(Q_0) = \min\{k(P) : P \in V(\gamma)\}$ and $k(Q_1) = \max\{k(P) : P \in V(\gamma)\}$. Assume now that the conclusion fails. Then by Lemma 34 we must have

$$k(Q_0) = \min k \leq 0.$$

Furthermore, k must be monotone on the two subintervals of $V(\gamma)$ that have Q_0 and Q_1 as endpoints.

Let Δ be the smallest closed disk that contains γ and let r denote its radius. Put $E = V(\gamma) \bigcap \partial\Delta$. Then $V(\gamma) \neq E$ by the assumptions on γ. We

remark that for all $P \in E$ the interior angle of γ at P is strictly less than π. From Lemma 35 it follows that $k(P) \geq \frac{1}{r}$ for all $P \in E$. Moreover, if $P \in E$ and if at least one immediate neighbor of P does not belong to E, then $k(P) > \frac{1}{r}$.

We now claim that E is not connected. To show the claim, we assume that E is an interval. Let N be the number of points in E. We must have $N \geq 2$. If $N = 2$ let $E = \{A, B\}$. Then A, B must lie on a diameter of Δ so that $|A - B| = 2r$. Since $k(A) > \frac{1}{r}$ by the above reasoning, it follows that A, B lie on a circle of radius strictly less than r, which is impossible. If $N > 2$ let $E_0 = \{P \in E : P \text{ is not an endpoint of } E\}$, then $E_0 \neq \emptyset$ and $k(P) = \frac{1}{r}$ for all $P \in E_0$. Since $k(P) > \frac{1}{r}$ for $P \in E \setminus E_0$ and $Q_0 \notin E$ but $k(Q_0) \leq 0$, it follows that k has at least two local minima, which again is incompatible with our assumption. We have therefore established that E is not connected.

Let $\mathcal{F}$ be the collection of maximal intervals that are contained in $V(\gamma) \backslash E$. Since E is not connected, $\mathcal{F}$ must contain at least two intervals. Set $F = \{P \in V(\gamma) : k(P) \leq 0\}$. It follows from the monotonicity properties of k that F is connected. Since $E \bigcap F = \emptyset$, there must exist a $J \in \mathcal{F}$ such that $F \bigcap J = \emptyset$, i.e., $k(P) > 0$ for all $P \in J$.

Let $\hat{J}$ be the union of J and its two immediate neighbors. Setting $J^* = \hat{J} \setminus J$, we have $J^* \subset E$ and $k(P) > \frac{1}{r}$ for all $P \in J^*$. Moreover, J^* consists of two distinct vertices of γ.

Denote by Γ the polygonal sub-arc of γ that has $\hat{J}$ as its vertex set. Then Γ separates Δ into two closed domains; let U be that domain that contains $\Omega(\gamma)$. Let Γ^* be that sub-arc of ∂U that has J^* as its endpoints and is contained in $\partial\Delta$. Notice that $\partial U = \Gamma \bigcup \Gamma^*$. Since $k(P) > 0$ for all $P \in J$, it follows that U is convex.

Denote by l the length of Γ^*. We claim that $l \geq \pi r$. Noticing that $J \subset \text{Int}(\Delta)$, we see that if $l < \pi r$, then $\hat{J} \bigcup \Gamma^*$ is contained in a disk of radius $r_1 < r$. This is impossible since $\Omega(\gamma) \subset U$ and Δ is the smallest disk containing γ.

We now make a polygonal approximation of ∂U by selecting consecutive points $\{P_i\}_1^m$ on ∂U such that $P_i \in \partial U \backslash \Gamma$. In particular, $P_i \notin J^*$, $1 \leq i \leq m$. Set $V^* = \hat{J} \bigcup \{P_i : 1 \leq i \leq m\}$ and let γ^* be the convex polygon with vertex set V^*. Clearly γ^* is strictly convex. Let k^* denote the discrete curvature relative to γ^*.

From the monotonicity properties of k follows that

$$\min\{k(P) : P \in J\} \geq \min\{k(P) : P \in J^*\} > \frac{1}{r}.$$

Since $k(P) = k^*(P)$ for $P \in J$, we have that

$$k^*(P) > \frac{1}{r} \text{ for } P \in J.$$

For $P \in V^*$ we let $\alpha^*(P)$ be the interior angle of γ^* at P. Also we let $\omega^*(P)$ denote the closed disk whose boundary equals the circle of curvature of γ^* at P.

Since $l \geq \pi r$, we see that if P_1 and P_m have been selected sufficiently close to J^*, then $\alpha^*(P) > \pi/2$ for all $P \in J^*$. From Lemma 35 follows that $k^*(P) > \frac{1}{r}$ for $P \in J^*$. Summarizing, we have therefore obtained that

$$k^*(P) > \frac{1}{r} \text{ for all } P \in \hat{J}.$$

Let $S = \{P \in V^* : V^* \subset \omega^*(P)\}$. Clearly, $\Omega(\gamma) \subset \omega^*(P)$ for all $P \in S$ so we must have

$$k^*(P) \leq \frac{1}{r} \text{ for all } P \in S.$$

Since $\omega^*(P) = \Delta$ for all $P \in V^* \setminus \hat{J}$, we have that $S = V^* \setminus \hat{J}$ and $\{\omega^*(P) : P \in S\} = \{\Delta\}$. This contradicts however Theorem30, which says that $\{\omega^*(P) : P \in S\}$ consists of at least two distinct disks. This contradiction establishes the theorem. □

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