# DISCRETE DECOMPOSABILITY OF THE RESTRICTION OF $A_{\mathfrak{q}}(\lambda)$ WITH RESPECT TO REDUCTIVE SUBGROUPS II
## — micro-local analysis and asymptotic $K$-support

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Abstract. Let $G' \subset G$ be real reductive Lie groups. This paper offers a criterion on the triplet $(G, G', \pi)$ that the irreducible unitary representation $\pi$ of $G$ splits into a discrete sum of irreducible unitary representations of a subgroup $G'$ when restricted to $G'$, each of finite multiplicity. Furthermore, we shall give an upper estimate of the multiplicity of an irreducible unitary representation of $G'$ occurring in $\pi|_{G'}$.

## §1. Introduction

**1.1.** Our object of study is the restriction of an irreducible unitary representation of a real reductive Lie group with respect to its reductive subgroup with focus on a **discrete** decomposition with **finite** multiplicity.

**1.2.** Let $G$ be a real reductive linear Lie group, and $G'$ a subgroup that is reductive in $G$. We write $\widehat{G}$ (resp. $\widehat{G'}$) for the unitary dual of $G$ (resp. $G'$). If $\pi \in \widehat{G}$, then the restriction $\pi|_{G'}$ is decomposed into the direct integral of irreducible unitary representations of $G'$:

$$\pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_{G'}(\tau : \pi|_{G'})\ \tau\ d\mu(\tau),$$

where $d\mu$ is a Borel measure on $\widehat{G'}$ and $m_{G'}(\cdot : \pi|_{G'})\colon \widehat{G'} \to \mathbb{N} \cup \{\infty\}$ is the multiplicity defined almost everywhere with respect to $d\mu$.

We say that the restriction $\pi|_{G'}$ is $G'$*-admissible* if the measure $d\mu$ is discrete and if the multiplicity $m_{G'}(\tau : \pi|_{G'}) < \infty$ for all $\tau \in \widehat{G'}$. It is a famous result due to Harish-Chandra that this is always the case if $G'$ is a maximal compact subgroup of $G$. Some other examples of the admissible restriction $\pi|_{G'}$ were previously known in the setting where $\pi$ is a highest weight module such as the Segal-Shale-Weil representation (cf. [4]) and a holomorphic discrete series representation (cf. [22], [21],

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[7]) for suitable $G'$. In [13], we gave a sufficient condition for the $G'$-admissibility of the restriction $\pi|_{G'}$ under the assumption that

$$\begin{aligned} \pi \quad &: \text{ a Zuckerman-Vogan's derived functor module } A_{\mathfrak{q}}(\lambda), \\ (G, G') &: \text{ a reductive symmetric pair.} \end{aligned}$$

A distinguishing feature in this setting is that $G'$ is not compact and $A_{\mathfrak{q}}(\lambda)$ is not always a highest weight module. In this paper, the above setting is generalized to:

$$\begin{aligned} \pi &: \text{ any irreducible unitary representation of } G, \\ G' &: \text{ any closed subgroup that is reductive in } G. \end{aligned}$$

Furthermore, we give an upper estimate of the multiplicity $m_{G'}(\tau : \pi|_{G'})$.

Our approach, based on micro-local analysis, is different and simpler in the special setting that was treated in the first part [13], and only a complete replacement of the machinery gave the base for such a generalization. Conversely, a necessary condition for discrete decomposability will be discussed in the third part [16] based on the study of the associated varieties by purely algebraic methods.

**1.3.** Our main results (see Theorem 2.9) are interesting not only from representation theoretic view points (e.g. leading us to an analog of Howe's dual pair correspondence of the minimal unipotent representation of $SO(p, q)$ [19]), but also give new applications in other branches of mathematics such as:
1) Vanishing theorem of modular symbols defined by the arithmetic quotients of Riemannian symmetric spaces, which especially solves affirmatively a conjecture of T. Oda in connection with the theory of automorphic forms [18].
2) Construction of new discrete series representations for non-symmetric pseudo-Riemannian homogeneous manifolds of reductive type [15].
3) The finite dimensionality theorem of the space of the global solutions of certain system of partial differential equations (e.g. a generalization of the Gauss-Aomoto-Gelfand hypergeometric equations to higher order) by using a (generalized) Penrose construction (see [24], §7).

**1.4.** Our idea goes back to fundamental results ([10], [5]) of Kashiwara, Vergne and Howe in the late '70s, which are based on micro-local analysis (e.g. [23], [6], [9]). The novelty of our work is to establish a framework of "the admissible restriction" with a number of new examples of interest, which rely on the breakthrough in the theory of the unitary dual in the '80s (e.g. [27], [28], [31], [2]).

**1.5.** The main results of this paper were announced in [14]. I am grateful to R. Howe and M. Kashiwara for helpful discussions. I also expresses my hearty thanks to the the staff and the organizers, in particular, M. Flensted-Jensen and G. 'Olafsson, of the special year on "Analysis on Lie groups" at the Mittag-Leffler Institute, where most part of the manuscript was prepared.

## §2. Asymptotic $K$-support and the restriction of representations.

**2.1.** In this section, building on the work of Kashiwara and Vergne [10] (see also [5]), we study the restriction of an infinite dimensional representation with respect to a reductive subgroup from the micro-local point of view. Furthermore, we employ the minimal $K$-type theory of Vogan [25] to obtain an upper estimate of the multiplicity of each irreducible component occurring in the restriction formula. The main result is Theorem 2.9.

**2.2.** Let $K$ be a Lie group with Lie algebra $\mathfrak{k}_0$. We give a trivialization of the tangent bundle $TK$ by the right translation:

$$K \times \mathfrak{k}_0 \xrightarrow{\sim} TK, \quad (x, X) \mapsto \frac{d}{dt}|_{t=0} \exp(tX)x.$$

Then the cotangent bundle $T^*K$ is naturally identified with $K \times \mathfrak{k}_0^*$. With these identifications, the translation $L_a R_b \colon K \to K,\ x \mapsto axb\ (a, b \in K)$ induces bundle isomorphisms:

$$\begin{aligned} (L_a R_b)_* \colon TK \to TK, &\quad (x, X) \mapsto (axb, \mathrm{Ad}(a)X), \\ (L_a R_b)^* \colon T^*K \to T^*K, &\quad (axb, \lambda) \mapsto (x, \mathrm{Ad}^*(a)^{-1}\lambda). \end{aligned}$$

**2.3.** Suppose that $K$ is a connected compact Lie group. We write $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbb{R}} \mathbb{C}$ for the complexification of $\mathfrak{k}_0$. We fix a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$, and a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$. We write $\bar{C}$ ($\subset \sqrt{-1}\mathfrak{t}_0^*$) for the dominant Weyl chamber, $\Lambda$ ($\subset \sqrt{-1}\mathfrak{t}_0^*$) for the weight lattice and put

$$\Lambda_+ := \Lambda \cap \bar{C}.$$

We fix an $\mathrm{Ad}^*(K)$-invariant inner product on $\sqrt{-1}\mathfrak{k}_0^*$, and then regard $\sqrt{-1}\mathfrak{t}_0^*$ as a subspace of $\sqrt{-1}\mathfrak{k}_0^*$. We denote by $|\lambda|$ the induced norm for $\lambda \in \sqrt{-1}\mathfrak{t}_0^*$.

The Cartan-Weyl theory classifies the equivalence classes of finite dimensional irreducible representations of $K$ by means of highest weight vectors. This map will be denoted by

$$\Lambda_+ \xrightarrow{\sim} \widehat{K}, \quad \lambda \mapsto (\tau_\lambda, F(K, \lambda)).$$

The Peter-Weyl theorem gives the irreducible decomposition of the regular representation $L^2(K)$ as $K \times K$-modules:

$$L^2(K) \simeq \sum_{\lambda \in \Lambda_+}^{\oplus} F(K, \lambda) \boxtimes F(K, \lambda)^*.$$

The Paley-Wiener type theorem characterizes functions $\varphi$ on $K$ in terms of the development $\varphi = \sum_{\lambda \in \Lambda_+} \widehat{\varphi}_\lambda$ with $\widehat{\varphi}_\lambda \in F(K, \lambda) \boxtimes F(K, \lambda)^*$ as follows:

$$\varphi \in \mathcal{B}(K) \Leftrightarrow \sup_{\lambda \in \Lambda_+} \|\widehat{\varphi}_\lambda\|_{L^2(K)} \exp(-\varepsilon|\lambda|) < \infty \quad \text{for any } \varepsilon > 0, \tag{2.3.1}$$

$$\varphi \in \mathcal{D}'(K) \Leftrightarrow \sup_{\lambda \in \Lambda_+} \|\widehat{\varphi}_\lambda\|_{L^2(K)} (1 + |\lambda|^n)^{-1} < \infty \quad \text{for some } n > 0, \tag{2.3.2}$$

$$\varphi \in C^\infty(K) \Leftrightarrow \sup_{\lambda \in \Lambda_+} \|\widehat{\varphi}_\lambda\|_{L^2(K)} (1 + |\lambda|^n) < \infty \quad \text{for any } n > 0, \tag{2.3.3}$$

$$\varphi \in \mathcal{A}(K) \Leftrightarrow \sup_{\lambda \in \Lambda_+} \|\widehat{\varphi}_\lambda\|_{L^2(K)} \exp(\varepsilon|\lambda|) < \infty \quad \text{for some } \varepsilon > 0. \tag{2.3.4}$$

Here, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}'$ denote the sheaf of analytic functions, (Sato's) hyperfunctions, and (Schwartz's) distributions, respectively.

**2.4.** The singularity spectrum (or singular spectrum) $\mathrm{S.S.}\,\varphi$ of a hyperfunction $\varphi \in \mathcal{B}(K)$ is a closed cone in $\sqrt{-1}T^*K$. The books [8], [9] have been an indispensable guide into the analytic background. We need:

**Fact 2.4.** (Theorem 4.5 in [10]). *Let $\varphi = \sum_{\lambda \in \Lambda_+} \widehat{\varphi}_\lambda \in \mathcal{B}(K)$ and let $Y$ be a closed cone in $\sqrt{-1}\mathfrak{t}_0^*$ which is contained in $\bar{C}$. Then the following conditions are equivalent:*

i) $\mathrm{S.S.}\,\varphi \subset K \times (-\operatorname{Ad}^*(K) \cdot Y) \qquad (\subset \sqrt{-1}T^*K \simeq K \times \sqrt{-1}\mathfrak{t}_0^*)$.

ii) *If a closed cone $Y' \subset \bar{C}$ satisfies $Y \cap Y' = \emptyset$, then there exist constants $R_{Y'} > 0$ and $\varepsilon_{Y'} > 0$ such that $\|\widehat{\varphi}_\lambda\|_{L^2(K)} \leq R_{Y'} \exp(-\varepsilon_{Y'}|\lambda|)$ for any $\lambda \in \Lambda_+ \cap Y'$.*

*Here, a closed cone $Y$ in $\mathbb{R}^n$ means that $Y$ is a closed subset of $\mathbb{R}^n \setminus \{0\}$ such that $\lambda Y \subset Y$ for any $\lambda > 0$.*

**2.5.** Let $S$ be a subset of a real vector space $\mathbb{R}^n$. Following [9], Definition 2.4.3, we define a closed cone $S\infty$ in $\mathbb{R}^n \setminus \{0\}$ by

$$S\infty := \{y \in \mathbb{R}^n \setminus \{0\} : \text{ there exists a sequence } (y_n, \varepsilon_n) \in S \times \mathbb{R}_+ \text{ such that } \lim_{n\to\infty} \varepsilon_n y_n = y \text{ and } \lim_{n\to\infty} \varepsilon_n = 0\}.$$

The following lemma is straightforward.

**Lemma 2.5.** *Let $S$ be a subset of a real vector space $\mathbb{R}^n$, and $Y$ a closed cone in $\mathbb{R}^n$. Then the following two conditions are equivalent.*

i) *$S\infty \cap Y = \emptyset$.*

ii) *There exists an open cone $V$ containing $Y$ such that $S \cap V$ is relatively compact.*

**2.6.** There are several equivalent definitions of the singularity spectrum of a hyperfunction (see [3] and references therein). One of them is given by the support of the Fourier transform, which is also the idea of the analytic wave front set for distributions (e.g. [6]). That is, the Fourier transform $\widetilde{f}(x) := \int_{\mathbb{R}^n} f(y) \exp(\sqrt{-1}\langle x, y\rangle)\, dy$ of a function $f \in L^1_{loc}(\mathbb{R}^n)$ with the infra-exponential growth condition:

$$\sup_{y \in \mathbb{R}^n} |f(y)| \exp(-\varepsilon|y|) < \infty \text{ for any } \varepsilon > 0$$

has the singularity spectrum estimated by

$$\mathrm{S.S.}\,\widetilde{f} \subset \mathbb{R}^n \times \sqrt{-1}(\operatorname{Supp} f)\infty \quad (\subset \sqrt{-1}\,T^*\mathbb{R}^n).$$

Similar estimates hold for a hyperfunction on a compact Lie group $K$: Given $\varphi = \sum_{\lambda \in \Lambda_+} \widehat{\varphi}_\lambda \in \mathcal{B}(K)$ where $\widehat{\varphi}_\lambda \in F(K,\lambda) \boxtimes F(K,\lambda)^*$, we define

$$\operatorname{Supp}(\widehat{\varphi}) := \{\lambda \in \Lambda_+ : \widehat{\varphi}_\lambda \neq 0\} \subset \sqrt{-1}\mathfrak{t}_0^*. \tag{2.6.1}$$

It seems that the following Lemma 2.6 (1) and Proposition 2.7 (1) are implicit in [10], but we prove them for the sake of completeness.

**Lemma 2.6.** 1) *Retain the above notation. We have:*

$$\mathrm{S.S.}\,\varphi \subset K \times (-\operatorname{Ad}^*(K)\cdot(\operatorname{Supp}(\widehat{\varphi})\infty)) \qquad (\subset \sqrt{-1}T^*K \simeq K \times \sqrt{-1}\mathfrak{k}_0^*).$$

2) *Furthermore, we assume* $\|\widehat{\varphi}_\lambda\|_{L^2(K)} \in \mathbb{N}$ *for any* $\lambda \in \Lambda_+$. *If* $Y \subset \bar{C}$ *is a closed cone such that*

$$\mathrm{S.S.}\,\varphi \subset K \times (-\operatorname{Ad}^*(K)\cdot Y),$$

*then we have* $\operatorname{Supp}(\widehat{\varphi})\infty \subset Y$.

*Proof.* 1) We define a closed cone by $Y := \operatorname{Supp}(\widehat{\varphi})\infty \subset \sqrt{-1}\mathfrak{t}_0^*$. Let $Y'$ be an arbitrary closed cone of $\sqrt{-1}\mathfrak{t}_0^*$ such that $Y' \cap Y = \emptyset$. By Lemma 2.5 (i) $\Rightarrow$ (ii), we find an open cone $V$ containing $Y'$ such that $V \cap \operatorname{Supp}(\widehat{\varphi})$ is relatively compact. In particular, $Y' \cap \operatorname{Supp}(\widehat{\varphi})$ is a finite set because $\operatorname{Supp}(\widehat{\varphi})$ $(\subset \Lambda_+)$ is discrete. Hence there exist $R_{Y'} > 0$ and $\varepsilon_{Y'} > 0$ such that $\|\widehat{\varphi}_\lambda\|_{L^2(K)} \le R_{Y'} \exp(-\varepsilon_{Y'}|\lambda|)$ for any $\lambda \in \Lambda_+ \cap Y'$. Now the first statement follows from Fact 2.4 (ii) $\Rightarrow$ (i).
2) If $\operatorname{Supp}(\widehat{\varphi})\infty \not\subset Y$, then we could find closed cones $Y'$ and $Y''$ and an open cone $V$ such that

(2.6.2)(a) $\qquad \emptyset \neq Y'' \subset V \subset Y' \subset \bar{C},$

(2.6.2)(b) $\qquad Y'' \subset \operatorname{Supp}(\widehat{\varphi})\infty,$

(2.6.2)(c) $\qquad Y' \cap Y \subset \{0\}.$

It follows from Fact 2.4 (i) $\Rightarrow$ (ii) that there exist $R_{Y'} > 0$ and $\varepsilon_{Y'} > 0$ such that $\|\widehat{\varphi}_\lambda\|_{L^2(K)} \le R_{Y'} \exp(-\varepsilon_{Y'}|\lambda|)$ for any $\lambda \in \Lambda_+ \cap Y'$. Because $\|\widehat{\varphi}_\lambda\|_{L^2(K)} \in \mathbb{N}$, we have

$$\sharp\,(\operatorname{Supp}(\widehat{\varphi}) \cap Y') < \infty.$$

In particular, $\operatorname{Supp}(\widehat{\varphi}) \cap V$ is relatively compact. It follows from Lemma 2.5 (ii) $\Rightarrow$ (i) that $Y'' \cap \operatorname{Supp}(\widehat{\varphi})\infty = \emptyset$. This contradicts to (2.6.2)(b) and $Y'' \neq \emptyset$. Hence, we have $\operatorname{Supp}(\widehat{\varphi})\infty \subset Y$. $\square$

**2.7.** Suppose $(\pi, \mathcal{H})$ is a representation of $K$. The $K$-multiplicity is defined by

$$m_K(\cdot : \pi) \colon \widehat{K} \to \mathbb{N} \cup \{\infty\}, \quad \tau \mapsto m_K(\tau : \pi) := \dim \operatorname{Hom}_K(\tau, \pi).$$

The *asymptotic $K$-support* of $(\pi, \mathcal{H})$ (see [10]), denoted by $\mathrm{AS}_K(\pi)$, is defined by

(2.7.1)(a) $\qquad \operatorname{Supp}_K(\pi) := \{\lambda \in \Lambda_+ : m_K(\tau_\lambda : \pi) \neq 0\},$

(2.7.1)(b) $\qquad \mathrm{AS}_K(\pi) := \operatorname{Supp}_K(\pi)\infty.$

Then $\mathrm{AS}_K(\pi)$ is a closed cone contained in $\bar{C} \subset \sqrt{-1}\mathfrak{t}_0^*$ because $\operatorname{Supp}_K(\pi) \subset \Lambda_+$ and $\bar{C} = \Lambda_+\infty$.

We say that $m_K(\cdot : \pi)$ is of *infra-exponential growth*, if

(2.7.2) $\qquad \sup_{\lambda \in \Lambda_+} m_K(F(K,\lambda) : \pi)\exp(-\varepsilon|\lambda|) < \infty \quad$ for any $\varepsilon > 0$;

and of *polynomial growth*, if there exist constants $C$ and $N$ such that

(2.7.3) $\qquad m_K(F(K,\lambda) : \pi) \le C(1 + |\lambda|)^N \quad$ for any $\lambda \in \Lambda_+$.

**Proposition 2.7.** 1) *Assume that $m_K(\cdot : \pi)$ is of infra-exponential growth (resp. of polynomial growth). Then the $K$-character of $(\pi, \mathcal{H})$ given by*

$$\Theta_\pi^K := \sum_{\tau \in \widehat{K}} m_K(\tau : \pi) \operatorname{Trace} \tau \tag{2.7.4}$$

*is well-defined as a hyperfunction on $K$ with the singularity spectrum (resp. as a distribution on $K$ with the analytic wave front set):*

$$\mathrm{S.\,S.}\, \Theta_\pi^K \subset K \times (\mathrm{Ad}^*(K) \cdot \mathrm{AS}_K(\pi)), \quad (\subset \sqrt{-1}T^*K \simeq K \times \sqrt{-1}\mathfrak{k}_0^*).$$

2) *Conversely, with the same assumption in (1), if $W \subset \sqrt{-1}\bar{C}$ is a closed cone such that $\mathrm{S.\,S.}\, \Theta_\pi^K \subset K \times \mathrm{Ad}^*(K) \cdot W$, then $\mathrm{AS}_K(\pi) \subset W$.*

*Proof.* 1) The first statement follows from (2.3.1). We put $\varphi := \overline{\Theta_\pi^K}$ the complex conjugate of $\Theta_\pi^K$ and $\widehat{\varphi}_\lambda := m_K(\tau_\lambda : \pi)\overline{\operatorname{Trace} \tau_\lambda}$ for $\lambda \in \Lambda_+$. Then we have:

$$\begin{cases} \varphi = \displaystyle\sum_{\lambda \in \Lambda_+} \widehat{\varphi}_\lambda \in \mathcal{B}(K), \quad \text{with } \widehat{\varphi}_\lambda \in F(K, \lambda) \boxtimes F(K, \lambda)^*, \\ \operatorname{Supp}(\widehat{\varphi}) = \operatorname{Supp}(\widehat{\overline{\Theta_\pi^K}}) = \operatorname{Supp}_K(\pi), \quad \text{(see (2.6.1) and (2.7.1)(a) for definition).} \end{cases}$$

It follows from Lemma 2.6 (1) and definition (2.7.1)(b) that

$$\mathrm{S.\,S.}\, \Theta_\pi^K = -\,\mathrm{S.\,S.}\, \varphi \subset K \times \mathrm{Ad}^*(K)(\operatorname{Supp}(\widehat{\varphi})\infty) = K \times \mathrm{Ad}^*(K) \cdot \mathrm{AS}_K(\pi).$$

2) Because $m_K(\tau_\lambda : \pi) \in \mathbb{N}$, $\varphi$ satisfies the assumption of Lemma 2.6 (2). Therefore, it follows from $\mathrm{S.\,S.}\, \varphi = -\,\mathrm{S.\,S.}\, \Theta_\pi^K \subset K \times (-\,\mathrm{Ad}^*(K) \cdot W)$ that

$$\mathrm{AS}_K(\pi) = \operatorname{Supp}_K(\pi)\infty = \operatorname{Supp}(\widehat{\varphi})\infty \subset W.$$

Hence, we have proved the Proposition. $\square$

<u>**2.8.**</u> Suppose $H$ is a closed subgroup of $K$. We write $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}} \colon \mathfrak{k}_0^* \to \mathfrak{h}_0^*$ for the projection dual to the inclusion of Lie algebras $\mathfrak{h}_0 \hookrightarrow \mathfrak{k}_0$. The same notation $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}$ is used for the complexified map $\mathfrak{k}^* \to \mathfrak{h}^*$. Let $\mathfrak{h}_0^\perp := \operatorname{Ker}(\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}} \colon \mathfrak{k}_0^* \to \mathfrak{h}_0^*)$. For $\mu \in \widehat{H}$, we put a subset of $\Lambda_+ \simeq \widehat{K}$ by

$$\Lambda_+(\mu) := \{\lambda \in \Lambda_+ : [\tau_\lambda|_H : \mu] \neq 0\} = \operatorname{Supp}_K(L^2(K \times_H \mu)).$$

Here, $L^2(K \times_H \mu)$ denotes the Hilbert space consisting of $L^2$-sections for the $K$-equivariant vector bundle $K \times_H \mu \to K/H$ associated to $\mu \in \widehat{H}$. We note that we have naturally a unitary representation of $K$ on $L^2(K \times_H \mu)$ and that the left regular unitary representation of $K$ on $L^2(K)$ decomposes as follows:

$$L^2(K) \simeq \sum_{\mu \in \widehat{H}}^{\oplus} L^2(K \times_H \mu) \otimes \mu^* \simeq \sum_{\mu \in \widehat{H}}^{\oplus} \left( \sum_{\lambda \in \Lambda_+(\mu)}^{\oplus} \tau_\lambda \otimes \operatorname{Hom}_H(\tau_\lambda|_H, \mu) \right) \otimes \mu^*.$$

**Theorem 2.8.** *Let $H$ be a closed subgroup of a connected compact Lie group $K$. Let $\pi$ be a representation of $K$. We assume*

$$\mathrm{AS}_K(\pi) \cap \sqrt{-1}\,\mathrm{Ad}^*(K)\mathfrak{h}_0^{\perp} = \{0\}. \tag{2.8.1}$$

1) *Then we have*

$$\sharp\,(\Lambda_+(\mu) \cap \mathrm{Supp}_K(\pi)) < \infty, \tag{2.8.2}$$

*for each $\mu \in \widehat{H}$. If the $K$-multiplicity $m_K(\tau : \pi) < \infty$ for any $\tau \in \widehat{K}$, then the $H$-multiplicity $m_H(\mu : \pi|_H) := \dim \mathrm{Hom}_H(\mu, \pi|_H) < \infty$ for any $\mu \in \widehat{H}$.*
2) *If the $K$-multiplicity function $m_K(\cdot : \pi) : \widehat{K} \to \mathbb{N}$ is of infra-exponential growth (see (2.7.2)), then so is the $H$-multiplicity function $m_H(\cdot : \pi|_H) \colon \widehat{H} \to \mathbb{N}$. Furthermore, the restriction of $\Theta_\pi^K$ to an analytic submanifold $H \hookrightarrow K$ is well-defined as a hyperfunction, and the resulting hyperfunction $\Theta_\pi^K|_H \in \mathcal{B}(H)$ coincides with*

$$\Theta^H_{\pi|_H} := \sum_{\mu \in \widehat{H}} m_H(\mu : \pi|_H)\,\mathrm{Trace}\,\mu \ \in \mathcal{B}(H).$$

*In particular, $\Theta^H_{\pi|_H}$ has the singularity spectrum estimated by*

$$\mathrm{S.\,S.}(\Theta^H_{\pi|_H}) \subset H \times \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}\,(\mathrm{Ad}^*(K)\,\mathrm{AS}_K(\pi)) \quad (\subset \sqrt{-1}T^*H \simeq H \times \sqrt{-1}\mathfrak{h}_0^*).$$

3) *If the $K$-multiplicity function $m_K(\cdot : \pi) \colon \widehat{K} \to \mathbb{N}$ is of polynomial growth, then so is the $H$-multiplicity function $m_H(\cdot : \pi|_H) \colon \widehat{H} \to \mathbb{N}$.*

*Proof.* 1) Because $\mathrm{Ad}^*(K)\mathfrak{h}_0^{\perp}$ is a closed cone satisfying (2.8.1), we can find a closed cone $T$ in $\sqrt{-1}\mathfrak{t}_0^*$ such that

$$\mathrm{AS}_K(\pi) \Subset T \subset \bar{C}, \tag{2.8.3(a)}$$

$$T \cap \sqrt{-1}\,\mathrm{Ad}^*(K)\mathfrak{h}_0^{\perp} \subset \{0\}. \tag{2.8.3(b)}$$

We write $\complement(\mathring{T})$ for the complement in $\sqrt{-1}\mathfrak{t}_0^*$ of the interior set $\mathring{T}$ of $T$. Because $\mathrm{AS}_K(\pi) \cap \complement(\mathring{T}) = \mathrm{Supp}_K(\pi)\infty \cap \complement(\mathring{T}) = \emptyset$, there exists an open cone $V$ containing $\complement(\mathring{T})$ such that $\mathrm{Supp}_K(\pi) \cap V$ is relatively compact by Lemma 2.5 (i)$\Rightarrow$(ii). Because $V \supset \complement(\mathring{T}) \supset \complement T$ and because $\mathrm{Supp}_K(\pi) \subset \Lambda_+$ is discrete, we conclude that

$$\sharp\,(\mathrm{Supp}_K(\pi) \setminus T) < \infty. \tag{2.8.4}$$

If $\varphi = \sum_{\lambda \in \Lambda_+(\mu)} \widehat{\varphi}_\lambda \in L^2(K \times_H \mu)$, then $\varphi$ is regarded as an $L^2$-function on $K$ with value in the representation space of $\mu\,(\in \widehat{H})$ satisfying

$$\varphi(kh) = \mu(h)^{-1}\varphi(k) \qquad (k \in K,\, h \in H).$$

In particular,

$$\mathrm{S.S.}\,\varphi \subset \{(k,\xi) : \mathrm{Ad}^*(k)^{-1}\xi \in \sqrt{-1}\mathfrak{h}_0^{\perp}\} \subset K \times \sqrt{-1}\,\mathrm{Ad}^*(K)\mathfrak{h}_0^{\perp}.$$

It follows from (2.8.3)(b) and from Fact 2.4 (i)⇒(ii) that there exist constants $R_T > 0$ and $\varepsilon_T > 0$ such that

$$\|\widehat{\varphi}_\lambda\|_{L^2} \le R_T \exp(-\varepsilon_T|\lambda|) \quad \text{for any } \lambda \in \Lambda_+ \cap T. \tag{2.8.5}$$

In order that an estimate of the form (2.8.5) holds for any $\varphi = \sum_{\lambda\in\Lambda_+(\mu)} \widehat{\varphi}_\lambda \in L^2(K \times_H \mu)$, we must have $\sharp\,(\Lambda_+(\mu) \cap T) < \infty$. Then, combining with (2.8.4) we have proved (2.8.2).

Assume furthermore that $m_K(\tau : \pi) < \infty$ for any $\tau \in \widehat{K}$. By definition, $m_K(\tau_\lambda : \pi) \neq 0$ iff $\lambda \in \mathrm{Supp}_K(\pi)$ and $[\tau_\lambda|_H : \mu] \neq 0$ iff $\lambda \in \Lambda_+(\mu)$. Because

$$m_H(\mu : \pi|_H) = \sum_{\lambda\in\Lambda_+} m_K(\tau_\lambda : \pi)\,[\tau_\lambda|_H : \mu] = \sum_{\lambda\in\mathrm{Supp}_K(\pi)\cap\Lambda_+(\mu)} m_K(\tau_\lambda : \pi)\,[\tau_\lambda|_H : \mu] \tag{2.8.6}$$

is a finite sum by (2.8.2), we have $m_H(\mu : \pi|_H) < \infty$ for any $\mu \in \widehat{H}$.

2) The embedding $\iota\colon H \hookrightarrow K$ defines the conormal bundle $T_H^*K \to H$ by

$$T_H^*K := \mathrm{Ker}(\iota^*\colon T^*K|_H \to T^*H) \simeq \mathrm{Ker}\left((H \times \mathfrak{k}_0^*) \to (H \times \mathfrak{h}_0^*)\right) \simeq H \times \mathfrak{h}_0^{\perp}.$$

Then the natural projection

$$\rho\colon (H \times_K \sqrt{-1}T^*K) \setminus \sqrt{-1}T_H^*K \to \sqrt{-1}T^*H, \quad (h, \sqrt{-1}\xi) \mapsto (h, \sqrt{-1}\iota^*(\xi))$$

is identified with

$$\rho\colon H \times \sqrt{-1}(\mathfrak{k}_0^* \setminus \mathfrak{h}_0^{\perp}) \to H \times \sqrt{-1}\mathfrak{h}_0^*, \quad (h, \sqrt{-1}\xi) \mapsto (h, \sqrt{-1}\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}(\xi)),$$

respectively. It follows from Proposition 2.7 and (2.8.1) that

$$\begin{aligned}\mathrm{S.S.}\,\Theta_\pi^K \cap \sqrt{-1}T_H^*K &\subset (K \times \mathrm{Ad}^*(K)\cdot \mathrm{AS}_K(\pi)) \cap \left(H \times \sqrt{-1}\mathfrak{h}_0^{\perp}\right)\\ &\subset H \times \{0\} \quad (\subset \sqrt{-1}T^*K \simeq K \times \sqrt{-1}\mathfrak{k}_0^*).\end{aligned}$$

Therefore, it follows from [9], Theorem 3.1.3 that the restriction $\Theta_\pi^K|_H$ is well-defined as a hyperfunction on $H$ and that its singularity spectrum is estimated by

$$\begin{aligned}\mathrm{S.S.}\left(\Theta_\pi^K|_H\right) &\subset \rho\left((H \times_K \sqrt{-1}T^*K) \cap \mathrm{S.S.}\,\Theta_\pi^K\right)\\ &\subset H \times \mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}\left(\mathrm{Ad}^*(K)\cdot\mathrm{AS}_K(\pi)\right) \quad (\subset \sqrt{-1}T^*H \simeq H \times \sqrt{-1}\mathfrak{h}_0^*).\end{aligned}$$

Because $\Theta_\pi^K$ is an $\mathrm{Ad}(K)$-invariant (especially, $\mathrm{Ad}(H)$-invariant) hyperfunction on $K$, the restriction $\Theta_\pi^K|_H$ is also an $\mathrm{Ad}(H)$-invariant hyperfunction on $H$. Therefore, there exists uniquely a function $a : \widehat{H} \to \mathbb{C}$ of infra-exponential growth such that

$$\Theta_\pi^K|_H = \sum_{\mu\in\widehat{H}} a(\mu)\,\mathrm{Trace}\,\mu,$$

where the right side converges in the weak topology of $\mathcal{B}(H) \simeq \mathcal{A}'(H)$. Hence, in order to complete the proof of the second statement, it suffices to show $a(\mu) = m_H(\mu : \pi|_H)$ for each $\mu \in \widehat{H}$, namely, to show

$$\int_H \Theta_\pi^K|_H(h)\, \overline{\operatorname{Trace}\mu(h)}\, dh = m_H(\mu : \pi|_H), \tag{2.8.7}$$

where the Haar measure on a compact group $H$ is normalized so that the volume of $H$ is 1. Let us prove (2.8.7). We fix $\mu \in \widehat{H}$ once and for all. We put

$$\Theta_1 := \sum_{\lambda \in \Lambda_+(\mu)} m_K(\tau_\lambda : \pi) \operatorname{Trace}\tau_\lambda \in \mathcal{B}(K), \tag{2.8.8)(a}$$

$$\Theta_2 := \sum_{\lambda \notin \Lambda_+(\mu)} m_K(\tau_\lambda : \pi) \operatorname{Trace}\tau_\lambda \in \mathcal{B}(K). \tag{2.8.8)(b}$$

Then we have $\Theta_\pi^K = \Theta_1 + \Theta_2$. In view of (2.8.2), the sum in (2.8.8)(a) is taken over the finite set $\Lambda_+(\mu) \cap \operatorname{Supp}_K(\pi)$. Therefore, $\Theta_1 \in \mathcal{A}(K)$ and we have

$$\int_H \Theta_1|_H(h)\, \overline{\operatorname{Trace}\mu(h)}\, dh = \sum_{\lambda \in \Lambda_+(\mu) \cap \operatorname{Supp}_K(\pi)} m_K(\tau_\lambda : \pi)\, [\tau_\lambda|_H : \mu],$$

from the orthogonal relation of characters of compact groups. By (2.8.6), the proof of (2.8.7) will be complete if we show

$$\int_H \Theta_2|_H(h)\, \overline{\operatorname{Trace}\mu(h)}\, dh = 0. \tag{2.8.7'}$$

We choose $\tau' \in \widehat{K}$ such that $[\tau'|_H : \mu] \neq 0$. We take a function $\varphi_0 \in \tau' \boxtimes (\tau')^* \subset \mathcal{A}(K)$ satisfying

$$\int_H \varphi_0(h) \overline{\operatorname{Trace}\mu(h)}\, dh = 1.$$

We set

$$\varphi(k) := \int_H \varphi_0(kh) \overline{\operatorname{Trace}\mu(h)}\, dh.$$

Then $\varphi \in \tau' \boxtimes (\tau')^* \subset \mathcal{A}(K)$ and $\varphi(kh) = \varphi(k) \operatorname{Trace}\mu(h)$ for $k \in K$ and $h \in H$.

We define an analytic map by

$$\Phi : K \times H \to K,\ (k,h) \mapsto kh.$$

With the isomorphisms in §2.2, $\Phi_{(k,h)*} : T_{(k,h)}(K \times H) \to T_{kh}K$ is given by the linear map $\mathfrak{k}_0 + \mathfrak{h}_0 \to \mathfrak{k}_0$, $(X, Y) \mapsto (X, \operatorname{Ad}(k)Y)$, and then the pullback $\Phi^*_{(k,h)} : T^*_{kh}K \to T^*_{(k,h)}(K \times H)$ is by $\mathfrak{k}_0^* \to \mathfrak{k}_0^* + \mathfrak{h}_0^*$, $\xi \mapsto (\xi, \operatorname{pr}_{\mathfrak{k}\to\mathfrak{h}}\left(\operatorname{Ad}^*(k)^{-1}\xi\right))$. Because $\Phi : K \times H \to K$ is a submersion, the pullback of hyperfunctions on $K$ is well-defined. We put

$$F := \Phi^*(\Theta_2 \overline{\varphi}) \in \mathcal{B}(K \times H).$$

In view of

$$\mathrm{S.\,S.}(\Theta_2\overline{\varphi}) \subset \mathrm{S.\,S.}(\Theta_2) = \mathrm{S.\,S.}(\Theta_\pi^K) \subset K \times \mathrm{Ad}^*(K)\cdot \mathrm{AS}_K(\pi),$$

we have by [9], Theorem 3.1.7

$$\mathrm{S.\,S.}\,F \subset \{(k,h;\xi,\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}(\mathrm{Ad}^*(k)^{-1}\xi)) : k\in K, h\in H, \xi \in \mathrm{Ad}^*(K)\cdot \mathrm{AS}_K(\pi)\}.$$

We put

$$V(k) := \int_H F(k,h)\,dh \in \mathcal{B}(K).$$

If $\xi \in \mathrm{Ad}^*(K)\cdot\mathrm{AS}_K(\pi)$ satisfies $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{h}}(\mathrm{Ad}^*(k)^{-1}\xi) = 0$ then $\xi = 0$ by (2.8.1). Therefore, we have

$$\mathrm{S.\,S.}\,F \cap \big(\sqrt{-1}T^*K \times (H\times\{0\})\big) = \emptyset. \tag{2.8.9}$$

It follows from (2.8.9) that $\mathrm{S.\,S.}\,V = \emptyset$ (see [9], Theorem 3.2.1), namely, $V$ is real analytic, and also that

$$V(e) = \int_H F(e,h)\,dh \ (= \int_H \Theta_2|_H(h)\,\overline{\mathrm{Trace}\,\mu(h)}\,dh).$$

Hence, (2.8.7′) will be proved if we show $V\equiv 0$. We shall show $V\equiv 0$ by proving $\int_K V(k)f(k)\,dk = 0$ for any $f\in\tau_1\boxtimes\tau_1^*$ $(\subset L^2(K))$ and for any $\tau_1\in\widehat{K}$. We fix such $f$ and $\tau_1$. Then

$$\begin{aligned}\int_K V(k)f(k)\,dk &= \int_K\int_H \Theta_2(kh)\overline{\varphi(kh)}f(k)\,dh\,dk\\ &= \int_H\int_K \Theta_2(kh)\overline{\varphi(kh)}f(k)\,dk\,dh.\end{aligned}$$

Because $\mathrm{S.\,S.}\,F\cap\big((K\times\{0\})\times\sqrt{-1}T^*H\big)=\emptyset$ and $f$ is real analytic, we have

$$\begin{aligned}\int_K \Theta_2(kh)\overline{\varphi(kh)}f(k)\,dk &= \int_K \Big(\Theta_2(kh)\overline{\varphi(kh)}f(k)\Big)|_K\,dk\\ &= \int_K \sum_{\tau\notin\Lambda_+(\mu)} m_K(\tau:\pi)\,\mathrm{Trace}\,\tau(kh)\overline{\varphi(kh)}f(k)\,dk,\end{aligned}$$

where the first integration is that for $\mathcal{B}(K\times H)$ along the fiber ([9], Theorem 3.2.1), while the second and third ones are for $\mathcal{B}(K)$ with $h\in H$ fixed and the resulting function is real analytic on $H$. Because the sum in the last formula converges in the weak topology of $\mathcal{B}(K)\simeq\mathcal{A}'(K)$ and because $\varphi\in\tau'\boxtimes(\tau')^*$ and and $f\in\tau_1\boxtimes\tau_1^*$, the above integral equals

$$= \sum_{\substack{\tau\notin\Lambda_+(\mu)\\ \tau\in\mathrm{Supp}_K(\tau_1^*\otimes\tau')}} \int_K m_K(\tau:\pi)\,\mathrm{Trace}\,\tau(kh)\overline{\varphi(kh)}f(k)\,dk.$$

Now that the sum is taken over a finite set, we have

$$
\begin{aligned}
&\int_K V(k) f(k)\, dk \\
=& \sum_{\substack{\tau \notin \Lambda_+(\mu) \\ \tau \in \mathrm{Supp}_K(\tau_1^* \otimes \tau')}} m_K(\tau : \pi) \int_K \left( \int_H \operatorname{Trace} \tau(kh) \overline{\operatorname{Trace} \mu(h)}\, dh \right) \overline{\varphi(k)} f(k)\, dk \\
=& 0.
\end{aligned}
$$

Here, the last equality follows from $[\tau|_H : \mu] = 0$ for any $\tau \notin \Lambda_+(\mu)$. Thus, we have proved $V \equiv 0$ and so (2.8.7$'$). Hence the proof of the second statement is complete.
3) The third statement is proved if we employ the analytic wave front set for Schwartz's distributions instead of the singularity spectrum. $\square$

*Remark.* The equation $\Theta_\pi^K|_H = \Theta_{\pi|_H}^H$ is true as a *formal character*. In fact, it follows from $(\operatorname{Trace} \tau)\,|_H = \sum_{\mu \in \widehat{H}} [\tau|_H : \mu]\, \mu$ that we have formally

$$
\begin{aligned}
\left( \sum_{\tau \in \widehat{K}} m_K(\tau : \pi) \operatorname{Trace} \tau \right) |_H &= \sum_{\tau \in \widehat{K}} m_K(\tau : \pi) \left( \sum_{\mu \in \widehat{H}} [\tau|_H : \mu] \operatorname{Trace} \mu \right) \\
&= \sum_{\mu \in \widehat{H}} \left( \sum_{\tau \in \widehat{K}} m_K(\tau : \pi) [\tau|_H : \mu] \operatorname{Trace} \mu \right).
\end{aligned}
$$

However, the equation $\Theta_\pi^K|_H = \Theta_{\pi|_H}^H$ in $\mathcal{B}(H)$ under the assumption (2.8.1) does not follow directly from a general theory of hyperfunctions (e.g. [9], [23]), as one observes the following: Let a sequence of analytic functions $\psi_j$ on $K$ converge to $\psi \in \mathcal{B}(K)$ in the weak topology of $\mathcal{B}(K) \simeq \mathcal{A}'(K)$. If $\mathrm{S.S.}\,\psi \cap \sqrt{-1} T_H^* K \subset \{0\}$ then the restriction $\psi|_H$ is a well-defined hyperfunction on $H$, but $\psi_j|_H$ may not converge to $\psi|_H$ in the weak topology of $\mathcal{B}(H) \simeq \mathcal{A}'(H)$. Likewise for distributions (cf. [6], Theorem 8.2.3 and Definition 8.2.2 (ii)).

**2.9.** We are ready to state our main theorem. Let $G$ be a real reductive Lie group, $K$ a maximal compact subgroup of $G$, and $\theta$ the corresponding Cartan involution of $G$. Suppose $G'$ is a $\theta$-stable closed subgroup of $G$ with finitely many connected components. Then $G'$ is also a reductive Lie group with a maximal compact subgroup $K' := G' \cap K$ and we have a Cartan decomposition $\mathfrak{g}_0' = \mathfrak{k}_0' + \mathfrak{p}_0'$. We fix a $\theta$-stable Cartan subalgebra $\mathfrak{j}_0'$ of $\mathfrak{g}_0'$. We put

$$\mathfrak{j}_{\mathbb{R}}' := \sqrt{-1}(\mathfrak{j}_0' \cap \mathfrak{k}_0') + (\mathfrak{j}_0' \cap \mathfrak{p}_0').$$

Let ${\mathfrak{j}_{\mathbb{R}}'}^* := \mathrm{Hom}_{\mathbb{R}}(\mathfrak{j}_{\mathbb{R}}', \mathbb{R})$ and $\mathfrak{j}'^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{j}, \mathbb{C})$. Then $\mathfrak{j}'^* = {\mathfrak{j}_{\mathbb{R}}'}^* \otimes \mathbb{C}$. Correspondingly, the map taking the real part is denoted by

$$\mathrm{Re} : \mathfrak{j}'^* \to {\mathfrak{j}_{\mathbb{R}}'}^*.$$

We fix a norm $|\cdot|$ on ${\mathfrak{j}_{\mathbb{R}}'}^*$ which is invariant under the Weyl group $W(\mathfrak{g}')$ of the root system $\Delta(\mathfrak{g}', \mathfrak{j}')$. For an irreducible representation $\tau$ of $G'$, we denote by $\chi_\tau \in \mathfrak{j}'^*$ the $Z(\mathfrak{g}')$-infinitesimal character of the underlying $(\mathfrak{g}', K')$-module of $\tau$ via the Harish-Chandra isomorphism $\mathrm{Hom}_{\mathbb{C}\text{-alg.}}(Z(\mathfrak{g}'), \mathbb{C}) \simeq \mathfrak{j}'^*/W(\mathfrak{g}')$.

**Theorem 2.9.** *Assume that $\pi \in \widehat{G}$ satisfies*

$$\mathrm{AS}_K(\pi) \cap \mathrm{Ad}^*(K)\sqrt{-1}\mathfrak{k}_0'^{\perp} = \{0\}. \tag{2.9.1}$$

*Then we have the following.*
1) *The restriction $\pi|_{K'}$ is $K'$-admissible. In particular, the restriction $\pi|_{G'}$ is $G'$-admissible. Namely, we have a unitary equivalence of $G'$-modules:*

$$\pi|_{G'} \simeq \sum_{\tau \in \widehat{G'}}^{\oplus} m_{G'}(\tau : \pi|_{G'})\ \tau \qquad \text{(a discrete Hilbert sum)},$$

*where $m_{G'}(\tau : \pi|_{G'}) := \dim \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) < \infty$ for each $\tau \in \widehat{G'}$.*
2) *If $\tau \in \widehat{G'}$ is a discrete summand of the restriction $\pi|_{G'}$, then we have an upper estimate of the asymptotic $K$-support of $\tau$ by*

$$\mathrm{AS}_{K'}(\tau|_{K'}) \subset \mathrm{pr}_{\mathfrak{k}\to\mathfrak{k}'}\left(\mathrm{Ad}^*(K)\,\mathrm{AS}_K(\pi)\right).$$

3) *There exist $A > 0$ and $N \geq 0$ such that the $G'$-multiplicity satisfies*

$$m_{G'}(\tau : \pi|_{G'}) \leq A(1 + |\operatorname{Re}\chi_\tau|)^N \qquad \text{for any } \tau \in \widehat{G'}. \tag{2.9.2}$$

We note that the choice of $\mathfrak{j}'$ and the norm affect (2.9.2) only by a constant multiple of $A$.

*Remark 2.10.* In Theorem 2.9 (3), $N = 0$ means that the multiplicity is uniformly bounded with respect to $\tau$. We note that this is not the case in general. We have recently proved sufficient conditions for the uniform boundedness of the multiplicity $m_{G'}(\tau : \pi|_{G'})$ in the following two cases (i) and (ii).
i) ([17]) $\pi$ is any highest weight module, and $(G, G')$ is a reductive symmetric pair.
ii) (in a forthcoming paper with T. Oshima) $\pi \in \widehat{G}$ is arbitrary and $(G, G')$ is a real form of $(SL(n,\mathbb{C}), GL(n-1,\mathbb{C}))$ or $(O(n,\mathbb{C}), O(n-1,\mathbb{C}))$. In this case, the multiplicity $m_{G'}(\tau : \pi|_{G'})$ is uniformly bounded with respect to both $\tau \in \widehat{G'}$ and $\pi \in \widehat{G}$.

<u>**2.11.**</u> *Proof of Theorem 2.9.* The subquotient theorem (ref. [32], Theorem 3.5.6) asserts that $\pi$ can be realized as a subquotient of some principal series representation of $G$. In particular,

$$m_K(\tau : \pi|_K) \leq m_K(\tau : L^2(K)) = \dim \tau$$

for any $\tau \in \widehat{K}$. Let $\delta_K \in \sqrt{-1}\mathfrak{t}_0^*$ be half the sum of elements in $\Delta^+(\mathfrak{k}, \mathfrak{t})$. By Weyl's dimension formula, we have

$$\dim \tau_\lambda = \frac{\prod_{\alpha \in \Delta^+(\mathfrak{k},\mathfrak{t})} \langle \lambda + \delta_K, \alpha \rangle}{\prod_{\alpha \in \Delta^+(\mathfrak{k},\mathfrak{t})} \langle \delta_K, \alpha \rangle}.$$

We note that

$$\sharp\Delta^+(\mathfrak{k},\mathfrak{t}) = \frac{1}{2}(\dim_{\mathbb{C}}\mathfrak{k} - \dim_{\mathbb{C}}\mathfrak{t}) = \frac{1}{2}(\dim K - \operatorname{rank} K)$$

Hence, there is a constant $C > 0$ such that

$$m_K(\tau_\lambda : \pi|_K) \le C\ (1 + |\lambda|)^{\frac{1}{2}(\dim K - \operatorname{rank} K)} \tag{2.11.1}$$

for any $\pi \in \widehat{G}$ and $\lambda \in \Lambda_+ \simeq \widehat{K}$. Thus, the multiplicity function $m_K(\cdot : \pi|_K) : \widehat{K} \to \mathbb{N}$ is of polynomial growth.

1) Theorem 2.8 shows that the restriction $\pi|_{K'}$ is $K'$-admissible. Because $K' \subset G'$, it follows from [13], Theorem 1.2 that the restriction $\pi|_{G'}$ is $G'$-admissible, too.

2) With the notation in Theorem 2.8,

$$\Theta^{K'}_{\pi|_{K'}} = \sum_{\mu \in \widehat{K'}} m_{K'}(\mu : \pi|_{K'}) \operatorname{Trace}\mu$$

is a well-defined hyperfunction on $K'$ with the singularity spectrum

$$\mathrm{S.S.}(\Theta^{K'}_{\pi|_{K'}}) \subset K' \times \operatorname{pr}_{\mathfrak{k}\to\mathfrak{k}'}(\operatorname{Ad}^*(K)\operatorname{AS}_K(\pi)).$$

Because $\|m_{K'}(\mu : \pi|_{K'}) \operatorname{Trace}\mu\|_{L^2(K')} = m_{K'}(\mu : \pi|_{K'}) \in \mathbb{N}$, we can apply Lemma 2.6 (2) to obtain

$$\operatorname{AS}_{K'}(\pi|_{K'}) = \operatorname{Supp}_{K'}(\pi|_{K'})\infty = \operatorname{Supp}(\widehat{\Theta^{K'}_{\pi|_{K'}}})\infty \subset \operatorname{pr}_{\mathfrak{k}\to\mathfrak{k}'}(\operatorname{Ad}^*(K)\operatorname{AS}_K(\pi)).$$

Since $\tau$ is a subrepresentation of $\pi$, we have $\operatorname{Supp}_{K'}(\tau|_{K'}) \subset \operatorname{Supp}_{K'}(\pi|_{K'})$. Therefore, $\operatorname{AS}_{K'}(\tau|_{K'}) \subset \operatorname{AS}_{K'}(\pi|_{K'})$. Hence, $\operatorname{AS}_{K'}(\tau|_{K'}) \subset \operatorname{pr}_{\mathfrak{k}\to\mathfrak{k}'}(\operatorname{Ad}^*(K)\operatorname{AS}_K(\pi))$.

3) We recall the minimal $K$-type theory due to Vogan [25]. We may assume that $\mathfrak{j}'$ is a fundamental Cartan subalgebra of $\mathfrak{g}'$, that is, $\mathfrak{t}'_0 := \mathfrak{j}'_0 \cap \mathfrak{k}'_0$ is a Cartan subalgebra of $\mathfrak{k}'_0$. Fix a positive system $\Delta^+(\mathfrak{k}', \mathfrak{t}')$. We write $\delta_{K'} \in \sqrt{-1}\mathfrak{t}'^*_0$ for half the sum of elements in $\Delta^+(\mathfrak{k}', \mathfrak{t}')$. A minimal $K$-type of a $(\mathfrak{g}', K')$-module $\tau$ is a $K'$-type $F(K', \mu)$ for which $|\mu + 2\delta_{K'}|$ is minimal among all $K'$-types occurring in $\tau$. It follows from [11], Theorem 10.26 that there exists a constant $C_1 (\ge 0)$ depending only on $\mathfrak{g}$ with the following property: if $\tau$ is an irreducible $(\mathfrak{g}', K')$-module with infinitesimal character $\chi_\tau \in \mathfrak{j}'^*$ and with minimal $K$-type $F(K', \mu_\tau) \in \widehat{K'}$, then

$$|\operatorname{Re}\chi_\tau| \ge |\mu_\tau + 2\delta_{K'}| - C_1 \ge |\mu_\tau| \ - C_1. \tag{2.11.2}$$

Then we have

$$m_{G'}(\tau, \pi|_{G'}) \le \dim \operatorname{Hom}_{K'}(F(K', \mu_\tau), \pi|_{K'}). \tag{2.11.3}$$

Because $m_K(\cdot : \pi|_K) \colon \widehat{K} \to \mathbb{N}$ is of polynomial growth, so is $m_{K'}(\cdot : \pi|_{K'}) \colon \widehat{K'} \to \mathbb{N}$ from Theorem 2.8 (3). Thus,

$$\dim \operatorname{Hom}_{K'}(F(K', \mu_\tau), \pi|_{K'}) \le C(1 + |\mu_\tau|)^N. \tag{2.11.4}$$

It follows from (2.11.2) that there exists $A > 0$ (e.g. $A = C(1 + C_1)^N$) such that

$$C(1 + |\mu_\tau|)^N \le A(1 + |\operatorname{Re}\chi_\tau|)^N. \tag{2.11.5}$$

Combining (2.11.3), (2.11.4) and (2.11.5), we have

$$m_{G'}(\tau, \pi|_{G'}) \le A(1 + |\operatorname{Re}\chi_\tau|)^N.$$

This is what we wanted to prove. $\square$

**Example 2.12.** If $G' = K$, then $K' = K$ and ${\mathfrak{k}'_0}^{\perp} = \{0\}$. Therefore, the assumption (2.9.1) is automatically satisfied. This case corresponds to a result due to Harish-Chandra asserting that $\pi|_K$ *is* $K$*-admissible for any* $\pi \in \widehat{G}$ (e.g. [32], Theorem 3.4.10). In this case, we can take $N = \frac{1}{2}(\dim K - \operatorname{rank} K)$ for the constant $N$ in Theorem 2.9 (see (2.11.1)).

**Example 2.13.** Suppose that $\mathfrak{k}'_0$ is an ideal of $\mathfrak{k}_0$, or equivalently, $K$ is locally isomorphic to the direct product of $K'$ and a compact normal subgroup $K''$. Then the assumption (2.9.1) is equivalent to

$$\operatorname{AS}_K(\pi) \cap \sqrt{-1}(\mathfrak{k}''_0)^* = \{0\}. \tag{2.9.1$'$}$$

Indeed, according to the direct sum decomposition $\mathfrak{k}_0 = \mathfrak{k}'_0 \oplus \mathfrak{k}''_0$, we have $\mathfrak{k}_0^* = (\mathfrak{k}'_0)^* \oplus (\mathfrak{k}''_0)^*$. Therefore, $\operatorname{Ad}^*(K)\sqrt{-1}(\mathfrak{k}'_0)^{\perp} = \operatorname{Ad}^*(K)\sqrt{-1}(\mathfrak{k}''_0)^* = \sqrt{-1}(\mathfrak{k}''_0)^*$.

**Example 2.14.** Suppose that $(K, K')$ is a symmetric pair defined by an involution $\sigma$ of $K$. Take a maximal abelian subspace $\mathfrak{a}_0$ in $\{X \in \mathfrak{k}_0 : \sigma X = -X\}$ and extend it to a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$. We choose compatible positive systems $\Delta^+(\mathfrak{k}, \mathfrak{t})$ and $\Sigma^+(\mathfrak{k}, \mathfrak{a})$. We regard $\mathfrak{a}_0^* \subset \mathfrak{t}_0^*$ according to the direct sum decomposition $\mathfrak{t}_0 = (\mathfrak{t}_0 \cap \mathfrak{k}'_0) \oplus \mathfrak{a}_0$. Put $\bar{C}_{\mathfrak{a}} := \bar{C} \cap \sqrt{-1}\mathfrak{a}_0^*$. Then we have $\bar{C} \cap \operatorname{Ad}^*(K)\sqrt{-1}{\mathfrak{k}'_0}^{\perp} = \bar{C}_{\mathfrak{a}}$. Hence the assumption (2.9.1) amounts to

$$\operatorname{AS}_K(\pi) \cap \bar{C}_{\mathfrak{a}} = \{0\} \tag{2.9.1$''$}$$

because $\operatorname{AS}_K(\pi) \subset \bar{C}$.

## §3. Computation of the asymptotic $K$-support

In this section we compute the asymptotic $K$-support $\operatorname{AS}_K(\pi)$ for typical irreducible unitary representations, namely,
1) $\pi = A_{\mathfrak{q}}(\lambda)$, Vogan-Zuckerman's derived functor module (§3.2).
2) $\pi$ is a unitary highest weight module (§3.3).
3) $\pi$ is a unipotent representation of $SO(2p, 2q)$ (§3.4).

<u>**3.1.**</u> We begin with the following:

**Lemma 3.1.** *Suppose* $\pi$ *is a* $K$*-module (possibly of infinite dimensional).*
1) *If* $\pi'$ *is a* $K$*-submodule of* $\pi$*, then* $\operatorname{AS}_K(\pi') \subset \operatorname{AS}_K(\pi)$.
2) *If* $\sigma$ *is a (non-zero) finite dimensional* $K$*-module, then* $\operatorname{AS}_K(\pi \otimes \sigma) = \operatorname{AS}_K(\pi)$.

*Proof.* 1) Obvious from the inclusion $\operatorname{Supp}_K(\pi') \subset \operatorname{Supp}_K(\pi)$.
2) Let $Q \subset \Lambda$ $(\subset \sqrt{-1}\mathfrak{t}_0^*)$ be the set of $\mathfrak{t}_0$-weights of the representation $(\sigma, U)$. As is well-known (e.g. [26], Proposition 3.2.12), if $\lambda \in \Lambda_+$, then any irreducible $K$-submodule of $F(K, \lambda) \otimes U$ is of the form $F(K, \lambda + \nu)$ for some $\nu \in Q$. Therefore, we have

$$\operatorname{Supp}_K(\pi \otimes \sigma) \subset \operatorname{Supp}_K(\pi) + Q := \{\mu + \nu \in \sqrt{-1}\mathfrak{t}_0^* : \mu \in \operatorname{Supp}_K(\pi), \nu \in Q\}.$$

Since $Q$ is a finite set, we have $(\operatorname{Supp}_K(\pi) + Q)\infty = \operatorname{Supp}_K(\pi)\infty$. Hence,

$$\operatorname{AS}_K(\pi \otimes \sigma) = \operatorname{Supp}_K(\pi \otimes \sigma)\infty \subset \operatorname{Supp}_K(\pi)\infty = \operatorname{AS}_K(\pi). \tag{3.1.1}$$

Similarly, we have $\mathrm{AS}_K(\pi \otimes \sigma \otimes \sigma^*) \subset \mathrm{AS}_K(\pi \otimes \sigma)$. Since $\pi$ is a $K$-submodule of $\pi \otimes \sigma \otimes \sigma^*$, we have $\mathrm{AS}_K(\pi) \subset \mathrm{AS}_K(\pi \otimes \sigma \otimes \sigma^*)$ by (1). Hence, we have

$$\mathrm{AS}_K(\pi) \subset \mathrm{AS}_K(\pi \otimes \sigma). \tag{3.1.2}$$

Now, (2) follows from (3.1.1) and (3.1.2). $\square$

**3.2.** First, we consider Zuckerman-Vogan's derived functor modules. We start with a brief review of them. Standard references are [26], [32], and [11].

Given $\lambda \in \bar{C}$ $(\subset \sqrt{-1}\mathfrak{t}_0^*)$, we define a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$ by $\mathfrak{q} \equiv \mathfrak{q}(\lambda) := \mathfrak{l} + \mathfrak{u}$, where $\mathfrak{l}$ and $\mathfrak{u}$ are stable under $\mathrm{ad}(\mathfrak{h})$ with weights given by

$$\Delta(\mathfrak{l}, \mathfrak{h}) := \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \alpha, \lambda \rangle = 0\},$$
$$\Delta(\mathfrak{u}, \mathfrak{h}) := \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \alpha, \lambda \rangle > 0\}.$$

Let $L := Z_G(\lambda)$, and $\widetilde{L}$ a metaplectic covering of $L$ defined by the character of $L$ acting on $\wedge^{\mathrm{top}}\mathfrak{u} \simeq \mathbb{C}_{2\rho(\mathfrak{u})}$. Here we write the character $\mathbb{C}_{2\rho(\mathfrak{u})}$ in an additive way. We note that $\mathfrak{l}$ is the complexified Lie algebra of $L$ (and also of $\widetilde{L}$). The elliptic orbit $\mathrm{Ad}(G)X \simeq G/L$ carries a $G$-invariant complex structure, with the canonical line bundle $\Omega := \wedge^{\mathrm{top}} T^*(G/L) \simeq G \times_L \mathbb{C}_{2\rho(\mathfrak{u})}$. Suppose $W$ is a finite dimensional metaplectic representation of $\widetilde{L}$. Then the $L$-module $W \otimes \mathbb{C}_{\rho(\mathfrak{u})}$ defines a $G$-homogeneous holomorphic vector bundle $\mathcal{W} := G \times_L (W \otimes \mathbb{C}_{\rho(\mathfrak{u})})$. As an algebraic analog of the Dolbeault cohomology group $H^j_{\bar\partial}(G/L, \mathcal{W})$ (see [33]), Zuckerman introduced the cohomological parabolic induction $\mathcal{R}^j_{\mathfrak{q}} \equiv \left(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}}\right)^j$ $(j \in \mathbb{N})$, which is a covariant functor from the category of metaplectic $(\mathfrak{l}, (L \cap K)^\sim)$-modules to that of $(\mathfrak{g}, K)$-modules. If the $Z(\mathfrak{l})$-infinitesimal character $\gamma \in \mathfrak{h}^*$ of an irreducible metaplectic representation $W$ of $\widetilde{L}$ lies in the *good range* in the sense of [29], Definition 2.5, namely, if $\mathrm{Re}\langle \gamma, \alpha \rangle > 0$ for any $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^c)$, then $\mathcal{R}^j_{\mathfrak{q}}(W) = 0$ for any $j \neq S := \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$ and $\mathcal{R}^S_{\mathfrak{q}}(W)$ is an irreducible $(\mathfrak{g}, K)$-module. Furthermore, if $\dim W = 1$, we sometimes write $A_{\mathfrak{q}}(\lambda)$ for $\mathcal{R}^S_{\mathfrak{q}}(W)$ where $\mathbb{C}_\lambda := W \otimes \mathbb{C}_{-\rho(\mathfrak{u})}$ is a character of $L$. Here, we follow the normalization in [28], Definition 6.20 for $\mathcal{R}^j_{\mathfrak{q}}$; and the one in [30], §5 for $A_{\mathfrak{q}}$. We define a closed cone (see [13]) by

$$\mathbb{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle := \left\{ \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t})} n_\beta \, \beta : \ n_\beta \geq 0 \right\} \quad (\subset \sqrt{-1}\mathfrak{t}_0^*). \tag{3.2.1}$$

**Example 3.2.** With notation as above, suppose $W$ is a finite dimensional metaplectic representation of $\widetilde{L}$. Then we have

$$\mathrm{AS}_K(\mathcal{R}^S_{\mathfrak{q}}(W)) \subset \mathbb{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle \cap \bar{C}.$$

*Proof of Example 3.2.* We write $\sigma_1 \underset{K}{\leq} \sigma_2$ if $\sigma_1$ and $\sigma_2$ are $K$-modules such that $\dim \mathrm{Hom}_K(\pi, \sigma_1) \leq \dim \mathrm{Hom}_K(\pi, \sigma_2)$ for any $\pi \in \widehat{K}$. Then, as is proved in [11], Theorem 5.35, we have the following estimate as $K$-modules:

$$\mathcal{R}^j_{\mathfrak{q}}(W)|_K \underset{K}{\leq} H^j_{\bar\partial}(K/L \cap K, S(\mathfrak{u} \cap \mathfrak{p}) \otimes W \otimes \mathbb{C}_{\rho(\mathfrak{u})}), \qquad j \in \mathbb{N}.$$

If $\nu - 2\rho(\mathfrak{u} \cap \mathfrak{k}) \in \Lambda_+$, then we have

$$H^S_{\bar\partial}(K/L \cap K, F(L \cap K, \nu)) = F(K, \nu - 2\rho(\mathfrak{u} \cap \mathfrak{k}))$$

from the Borel-Weil-Bott theorem. Therefore, if we define a finite set $Q$ by $Q := \{\delta + \rho(\mathfrak{u}) - 2\rho(\mathfrak{u} \cap \mathfrak{k}) : \delta$ is a highest weight of $(L \cap K)^\sim$ type occurring in $W\}$, then, by using the notation analogous to (3.2.1) ($\mathbb{R}_+$ replaced by $\mathbb{N}$), we have

$$\begin{aligned}\operatorname{Supp}_K(\mathcal{R}^S_{\mathfrak{q}}(W)) &\subset \operatorname{Supp}_K(H^S_{\bar\partial}(K/L \cap K, S(\mathfrak{u} \cap \mathfrak{p}) \otimes W \otimes \mathbb{C}_{\rho(\mathfrak{u})})) \\ &\subset (\mathbb{N}\langle \mathfrak{u} \cap \mathfrak{p}\rangle + Q) \cap \bar{C}\end{aligned}$$

because any irreducible representation of $L \cap K$ occurring in the symmetric tensor algebra $S(\mathfrak{u} \cap \mathfrak{p})$ is of the form $F(L \cap K, \nu)$ with $\nu \in \mathbb{N}\langle \mathfrak{u} \cap \mathfrak{p}\rangle$. Thus, we have $\operatorname{AS}_K(\mathcal{R}^S_{\mathfrak{q}}(W)) = \operatorname{Supp}_K(\mathcal{R}^S_{\mathfrak{q}}(W))\infty \subset \mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p}\rangle \cap \bar{C}$. $\square$

**3.3.** Next, we consider unitary highest weight modules.

Suppose that $G/K$ is an irreducible Hermitian symmetric space. Let $\mathfrak{c}(\mathfrak{k}_0)$ be the center of $\mathfrak{k}_0$. Then, it is known that there exists $Z \in \mathfrak{c}(\mathfrak{k}_0)$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

are 0, $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $\operatorname{ad}(Z)$. An irreducible $(\mathfrak{g}, K)$-module is said to be a *highest weight module* if there exists a non-zero vector annihilated by $\mathfrak{p}^+$.

**Example 3.3.** If $V$ is a unitary highest weight $(\mathfrak{g}, K)$-module, then

$$\operatorname{AS}_K(V) \subset \mathbb{R}_+\langle \mathfrak{p}^-\rangle \cap \bar{C}. \tag{3.3.1}$$

*Proof of Example 3.3.* Let $G_{\mathbb{C}}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}$, $P_-$ a maximal parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{k} + \mathfrak{p}^-$. We equip $G/K$ with a $G$-invariant complex structure through the Borel embedding:

$$G/K \hookrightarrow G_{\mathbb{C}}/P_-.$$

Then we can realize $V$ as a $(\mathfrak{g}, K)$-submodule of $\mathcal{O}(G \times_K U)$, where

$$U := V^{\mathfrak{p}^+} = \{v \in V_K : d\pi(Y)v = 0 \ \text{ for any } Y \in \mathfrak{p}^+\} \ (\neq \{0\})$$

is an irreducible representation of $K$. Since the $K$-structure of the underlying $(\mathfrak{g}, K)$-module of $\mathcal{O}(G \times_K U)$ is given by $\mathcal{O}(G \times_K U)_K \simeq S(\mathfrak{p}^-) \otimes U$, we have

$$\operatorname{AS}_K(V) \subset \operatorname{AS}_K(S(\mathfrak{p}^-) \otimes U) = \operatorname{AS}_K(S(\mathfrak{p}^-)) \subset \mathbb{R}_+\langle \mathfrak{p}^-\rangle \cap \bar{C}.$$

Here, we have used Lemma 3.1 (1) and (2). Thus the proof is completed. $\square$

*Remark.* Following [1], Definition 1.6, we say that a $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is *anti-holomorphic* if $\mathfrak{u} \cap \mathfrak{p} \subset \mathfrak{p}^-$. If $\mathfrak{q}$ is anti-holomorphic, then any irreducible $(\mathfrak{g}, K)$-subquotient $V$ of $\mathcal{R}^S_{\mathfrak{q}}(W)$ is a highest weight module for any finite dimensional metaplectic representation $W$ of $\widetilde{L}$ ([1], Lemma 1.7). In this case, Example 3.2 gives an estimate

$$\operatorname{AS}_K(V) \subset \mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p}\rangle \cap \bar{C} \quad (\subset \mathbb{R}_+\langle \mathfrak{p}^-\rangle \cap \bar{C}),$$

which is a refinement of (3.3.1).

**3.4.** Third, we give an example of a "unipotent representation". Similar computation works for other "small representations" of other reductive groups in an elementary way, especially when its $K$-type formula is of "ladder type".

**Example 3.4.** Let $G = SO_o(2p, 2q)$, for $p \geq 1$, $q \geq 1$. Choose a basis $\{f_i : 1 \leq i \leq p+q\}$ on $\sqrt{-1}\mathfrak{t}_0^*$ such that a positive root system of $\mathfrak{k}$ is given by

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{f_i \pm f_j : 1 \leq i < j \leq p, \quad \text{or } p+1 \leq i < j \leq p+q\}.$$

Kostant, Binegar and Zierau constructed an irreducible unitary representation $(\pi, \mathcal{H})$ of $G$ (also that of $SO_0(2p+1, 2q+1)$), which is "associated with" the minimal nilpotent coadjoint orbit [20], [2] (different constructions are also known by Torasso, Huang-Zhu, and [19]). This representation $(\pi, \mathcal{H})$ is not a highest (or lowest) weight module if $p > 1$ and $q > 1$, and has the $K$-structure of the so called "ladder" type:

$$\mathcal{H}_K \simeq \bigoplus_{\substack{a,b \in \mathbb{N} \\ a+p=b+q}} F(SO(2p), af_1) \boxtimes F(SO(2q), bf_{p+1}).$$

Thus, $\operatorname{Supp}_K(\pi) = \{af_1 + bf_{p+1} : a, b \in \mathbb{N}, a+p = b+q\}$ and so

$$\operatorname{AS}_K(\pi) = \mathbb{R}_+(f_1 + f_{p+1}).$$

If $G' = SO(2p, q') \times SO(q'')$ $(q' + q'' = 2q)$ then $\mathfrak{a} = \sum_{j=1}^{l} \mathbb{R} f_{p+j}$ with notation in Example 2.14, where $l := \min(q', q'')$. If $G' = U(p, q)$, then $\mathfrak{a}$ is of the form

$$\begin{aligned} &\{(x_1, x_1, x_2, x_2, \ldots, x_{[\frac{p}{2}]-1}, (-1)^{[\frac{p}{2}]} x_{[\frac{p}{2}]}, (0)) : x_i \in \mathbb{R}\} \\ \oplus &\{y_1, y_1, y_2, y_2, \ldots, y_{[\frac{q}{2}]-1}, (-1)^{[\frac{q}{2}]} y_{[\frac{q}{2}]}, (0)) : y_j \in \mathbb{R}\}. \end{aligned}$$

Therefore, both of the restrictions $\pi|_{SO(2p,q') \times SO(q'')}$ and $\pi|_{U(p,q)}$ are discretely decomposable. Explicit branching laws will be presented in a forthcoming paper joint with B. Ørsted, giving an analog of Howe's correspondence ([19]).

*Remark 3.5.* Theorem 2.9 involves two invariants, namely, $\operatorname{Ad}^*(K)\sqrt{-1}{\mathfrak{k}'_0}^{\perp}$ and $\operatorname{AS}_K(\pi)$, of which typical examples are computed explicitly in Examples 2.12, 2.13, 2.14 and in Examples 3.2, 3.3, 3.4, respectively. The combination of these examples gives rise to both new results and a unified treatment of previous results. We collect them for the convenience of the reader.

1) Example 2.12 corresponds to a result of Harish-Chandra asserting that any irreducible unitary representation of $G$ is $K$-admissible. The explicit branching law in the setting of the combination of Example 2.12 and Example 3.2 is called the generalized Blattner conjecture (theorem) proved by H. Hecht and W. Schmid (discrete series case) and by D. Vogan (general case) (see [26]).

2) The combination of Example 2.13 (applied to the case $\mathfrak{k}'_0$ is the center of $\mathfrak{k}_0$) and Example 3.3 (applied to holomorphic discrete series representations) gives a theorem of Martens [22] and of Lipsman and Wallach ([21], Theorem 4.2). The decomposition formula was studied by Jakobsen and Vergne [7].

3) The combination of Example 2.13 and Example 3.2 leads to [12], Proposition 4.1.3, which was used to determine the non-vanishing condition of unitary cohomologically induced representations $\mathcal{R}_{\mathfrak{q}}^S(\mathbb{C}_\lambda)$ with singular parameters $\lambda$.

4) The combination of Example 2.14 and Example 3.2 yields the main theorem of [13] (see Theorem 3.2 loc. cit.). In [16], Theorem 4.2, we prove that our condition is also necessary to be $K'$-admissible in this case.
5) A special case of Example 2.14 (that is, $(G, G') = (SO(2, 2n), SO(1, 2n))$) combined with Example 3.2 leads us to a very explicit formula of the middle Hodge component of certain modular symbols that are totally real and totally geodesic cycles in the arithmetic quotients of bounded symmetric domain of type IV [18].
6) The combination of Example 2.14 and Example 3.4 presents new examples of an analog of Howe's dual pair correspondence for minimal unipotent representations [19].

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