Tentamen i MAN410 Topologi, 050607, kl. 8.30-13.30. Inga hjälpmedel.

1. (10p) Let X be a topological space. A *subbase* for the topology on X is a family $\mathcal{S}$ of open subsets of X such that every open set can be written as a union of finite intersections of sets in $\mathcal{S}$.

   (a) Show that any family of subsets of a space X is a subbase for some topology on X.

   (b) Show that $\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ is a subbase for the usual topology on $\mathbb{E}^1$.

   (c) Let X and Y be topological spaces and let $\mathcal{S}$ be a subbase for the topology on Y. Show that $f : X \to Y$ is continuous iff $f^{-1}(A)$ is open for every $A \in \mathcal{S}$.

2. (6p) Show that if X is a topological space, Y is a subspace of X and Z is a subset of Y, then Z is compact as a subspace of X iff Z is compact as a subspace of Y. Then use this to show that if Y and Z are two subspaces of X then $Y \cap Z$ is compact in Y iff $Y \cap Z$ is compact in Z.

3. (12p) Consider a topological group G and let e denote the identity element.

   (a) Show that if U is an open neighborhood of e, then there exists an open neighborhood V of e such that $VV^{-1} \subseteq U$.

   (b) Assume that H is a subgroup of G such that when H is regarded as a subspace of G, H is a discrete space. Show that there exists an open neighborhood N of e such that hN and N are disjoint for every $h \in H \setminus \{e\}$. (Hence all the hN's, $h \in H$, are disjoint.)

   (c) Let H be as in (b) and let C be a closed subset of G. Show that $H \cap C$ is a finite set.

4. (10p) Let X and Y be homeomorphic topological spaces and let $h : X \to Y$ be a homeomorphism. Show that for any $A \subseteq X$, the spaces $X \setminus A$ and $Y \setminus f(A)$ are homeomorphic. Demonstrate that the converse is false by finding a topological space X and subspaces A and B such that A and B are homeomorphic, but $X \setminus A$ and $X \setminus B$ are not.

5. (12p)

(a) Show that $\mathbb{E}^1$ and $\mathbb{E}^2$ are not homeomorphic.

(b) Show that $\mathbb{E}^2$ and $\mathbb{E}^3$ are not homeomorphic.

(c) For $n \geq 2$ identify $\mathbb{E}^1$ with the first coordinate axis in $\mathbb{E}^n$. Show that $\mathbb{E}^n \setminus \mathbb{E}^1$ has the same homotopy type as $\mathbb{E}^{n-1} \setminus \{0\}$.

(d) Show that $\mathbb{E}^3$ and $\mathbb{E}^4$ are not homeomorphic.

Telefonvakt under tentamen är Iulia Pop som kan sökas på telefon 076-2721861.

Efter skrivtidens slut finns förslag till lösningar på kursens hemsida. Skrivningen beräknas vara färdigrättad fredagen den 17 juni. Vardagar efter kl 14.00 kan man få reda på resultat på telefon 7723509.

Lycka till! /Johan Jonasson

Lösningar

1. (a) Let $\mathcal{B}$ be the family of finite intersections of sets in $\mathcal{S}$. Then $X \in \mathcal{B}$ (the empty intersection), and by definition $\mathcal{B}$ is closed under finite intersections. Thus $\mathcal{B}$ is the base for some topology. This topology has $\mathcal{S}$ as a subbase.

   (b) Pick an open set U. Since U is open, one can for every $x \in U$ find a positive number δ_x such that $(x - \delta_x, x + \delta x) \subseteq U$. Put A_x for this interval. Then $A_x = (-\infty, x + \delta_x) \cap (x - \delta_x . \infty)$ and $U = \bigcup_{x \in U} A_x$.

   (c) Let $\mathcal{B}$ be as in (a). Since $\mathcal{B}$ is a base for the topology on Y it suffices to check that inverse images of sets in $\mathcal{B}$ are open. However since intersections and inverse images commute, this follows from the assumption.

2. Assume first that Z is compact in Y and let $\{U_i\}_{i \in I}$ be an open cover in X of Z. Then with $U_i' = U_i \cap Y$ the U_i''s are open in Y and cover Z. Thus there exists a finite subcover, $\{U_1', \ldots, U_n'\}$. Then $\{U_1, \ldots, U_n\}$ is a subfamily of the original cover and covers Z. Hence Z is compact in X.

   Conversely, let $\{U_i\}$ be an open cover in Y of Z. Then by the definition of the induced topology, we can write the U_i's as $V_i \cap Y$ where the V_i's are open in X. The V_i's cover Z so if Z is compact in X there is a finite subset of the V_i's that cover Z. Since Z is a subset of Y, the corresponding finite set of U_i's also cover Z.

For the second part, use the first part twice: $Y \cap Z$ is compact in Y iff $Y \cap Z$ is compact in X which in turn happens iff $Y \cap Z$ is compact in Z.

3. (a) By the continuity of the group operation there exists an open set W such that $WW \subseteq U$. By continuity of inverses, W^{-1} is open. Put $V = W \cap W^{-1}$. Then $V^{-1} = V$ so $VV^{-1} = VV \subseteq WW \subseteq U$.

   (b) Since H is discrete G has a neighborhood U of e such that U contains no other point of H. By (a) there is a neighborhood N of e such that $NN^{-1} \subseteq U$. Then if $h \in H \setminus \{e\}$, hN and N are disjoint, for if not there would exist elements n_1 and n_2 in N such that $hn_1 = n_2$, i.e. $h = n_2 n_1^{-1}$, so that $h \in NN^{-1}$ a contradiction.

   (c) This is wrong as it stands. For example put $G = \mathbb{R}$, $H = C = \mathbb{Z}$. The assumption should have been that C is compact. In this case let N be as in (b). Then for every $g \in G$, Ng contains at most one element of H, for if h_1 and h_2 are two elements of H in this set then $h_1 = n_1 g$ and $h_2 = n_2 g$, $n_1, n_2 \in N$, so that $h_1 h_2^{-1} = n_1 n_2^{-1} \in NN^{-1}$ so that $h_1 h_2^{-1} = e$.

   Now $\{Ng : g \in C\}$ is an open cover of C and can hence be reduced to a finite subcover. This subcover contains at most finitely many points of H, as desired.

4. The map $h|X \setminus A : X \setminus A \to Y \setminus h(A)$ is clearly bijective, so it needs to be checked that it is continuous and has continuous inverse. This is part (b) and (c) of the exam 040601.

   For the second part, let $A = \mathbb{E}^1 \setminus \mathbb{Z}$ and $B = \mathbb{E}^1 \setminus \{0, -1, 1-1/2, 1/2, -1/3, 1/3, \ldots\}$. These are homeomorphic via the homeomorphism $h(x) = 1/x$. However the complements are not homeomorphic as one is compact and the other is not.

5. If there had been a homeomorphism $h : \mathbb{E}^1 \to \mathbb{E}^2$, then the restriction to $\mathbb{E}^1 \setminus \{0\}$ would be a homeomorphism from a disconnected space to a connected space, a contradiction. Part (b) uses the same idea, with the only difference is that now the restriction to $\mathbb{E}^2 \setminus \{0\}$ is a homeomorphism from a space with fundamental group $\mathbb{Z}$ to a simply connected space.

For part (c) embed $\mathbb{E}^{n-1}$ in $\mathbb{E}^n$ and use the straight-line homotopy.

Part (d) is more difficult than what was intended. However, it is solvable by the methods learnt from the course. Observe as in (a) and (b) that if $\mathbb{E}^4$ and $\mathbb{E}^3$ were homeomorphic, then the same would go for the two sets with the origin removed. These spaces in turn have the same homotopy type as S^3 and S^2 respectively. It thus remains to show that S^2 and S^3 do not have the same homotopy type. It is easily seen that S^2 and S^3 are triangulable; put K and L respectively for such complexes. Now assume for contradiction that $|K|$ and $|L|$ have the same homotopy type. Then there exist continuous maps $f : |K| \to |L|$ and $g : |L| \to |K|$ such that fg and gf are homotopic to the two identity maps. We may assume that there exists a simplicial approximation s of f; otherwise barycentrically subdivide K as many times as required and use the simplicial approximation theorem.

Since s and f are homotopic, we get that sg is homotopic to 1_{S^3}. However by the definition of a simplicial approximation, the image of s, and hence of sg, is a union of simplexes of dimension at most 2. In particular sg is not onto. Therefore, if p is any point not in $sg(|L|)$, then sg is homotopic to the constant map at p via the normalized straight-line homotopy. Since sg is also homotopic to the identity map, we get that $|L|$, and hence also S^3, is contractible, a contradiction.