ODE, Mat460

What is an <u>ordinary differential equation</u>

- Differential equation: involves derivatives of functions
- Ordinary: only derivatives w.r.t. one variable.

<u>Examples</u> ($y = y(x)$, $\mathbb{R} \to \mathbb{R}$)

1) $y' = y$ ($y(x) = C_1 e^{x}$)

2) $y' + 2xy = 0$ ($y(x) = C_1 e^{-x^2}$)

3) $y'' + y = 0$ ($y(x) = A \sin x + B \cos x$)

4) $F(x, y, y') = 0$ ($F: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$)

4) is the most general ode of first order

5) $F(x, y, y', \dots, y^{(n)}) = 0$

a general n:th order equation:

<u>order</u> the highest derivative that occurs in the equation.

6) systems: $y \in \mathbb{R}^n$,

7) Explicit first order equations:

$$y' = f(x, y)$$

Something about contents of the course

1) first order ode's $y' = f(x,y)$.

Special cases that can be solved more or less explicitly.

2) systems

3) The initial value problem

$$\begin{cases} y'(x) = f(x, y(x)) & (x \geq x_0) \\ y(x_0) = y_0 \end{cases}$$

Existence, uniqueness, stability ...

4) boundary value problems

Ex: $\begin{cases} y + y'' = 0 & 0 < x < 1 \\ y(0) = y(1) = 0 \end{cases}$

$$y(x) = \sin(k\pi x)$$

general soln: $y = \sum a_k \sin(k\pi x)$

Generalizations of this

Sturm-Liouville theory

5) Linear systems,

stability

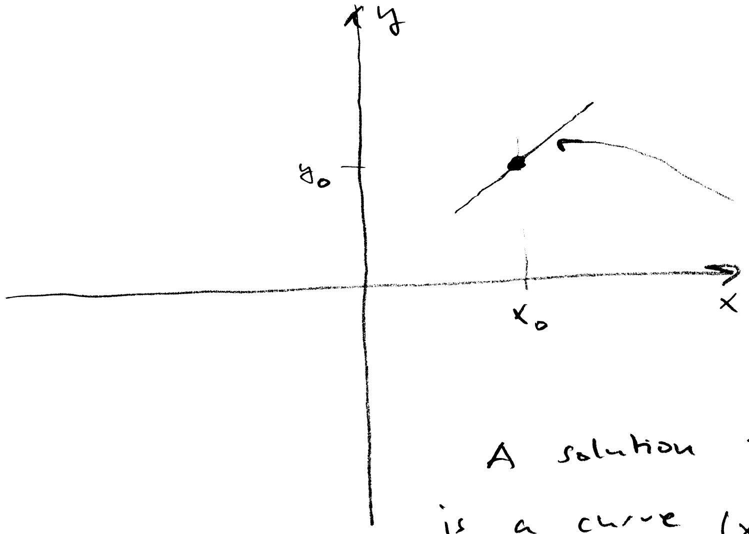
Dynamical systems.

Partial differential equations:

$$F(x, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots) = 0$$

$$x = (x_1, x_2, \dots x_n)$$

What is the solution of an ode?

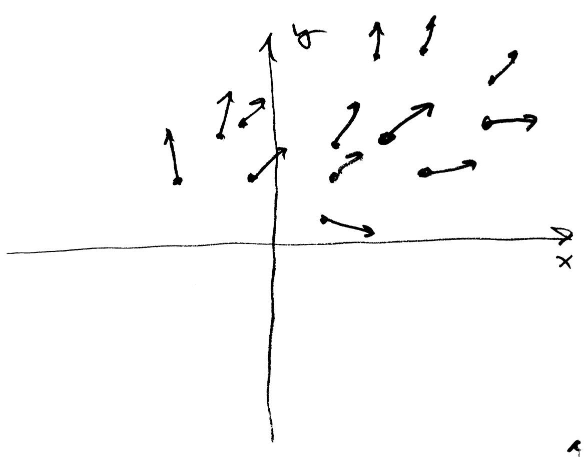


$y' = f(x,y)$

$y = y_0 + f'(x_0,y_0)\,(x - x_0)$

A solution to $y' = f(x,y)$ is a curve $(x, y(x)) \in \mathbb{R}^2$ $x \in I$ that is differentiable and such that at each point $(x_0, y(x_0))$ on the curve, it is tangent to the line $y = y_0 + f'(x_0,y_0)\,(x - x_0)$

A vector field



A vector field is a (cont) map $\mathbb{R}^2 \to \mathbb{R}^2$

Note $f(x,y): \mathbb{R}^2 \to \mathbb{R}$ corresponds to a vector field

$(x,y) \mapsto (1, f(x,y))$

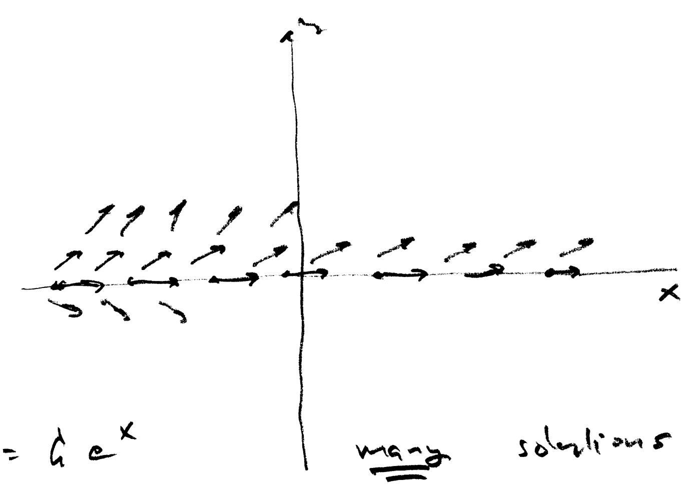
Ex: $y' = y$

$f(x,y) = y$,

vector field:

$(x,y) \mapsto (1, y)$

Solution. $y(x) = C e^x$



many solutions (unless we fix a point)

Special cases (that can be solved)

(Walter, ch. I)

d

$$\begin{cases} y'(x) = f(x) \\ y(x_0) = y_0 \end{cases} \Rightarrow \quad y(x) = \underset{\substack{\uparrow \\ \text{constant to fix initial data.}}}{y_0} + \int_{x_0}^{x} f(\xi)\, d\xi$$

Ex $\quad y'(x) = x^3 + \cos x$

$\Rightarrow \quad y(x) = \frac{1}{4}x^4 + \sin x + c$

$\boxed{\text{Fundamental theorem of calculus}}$

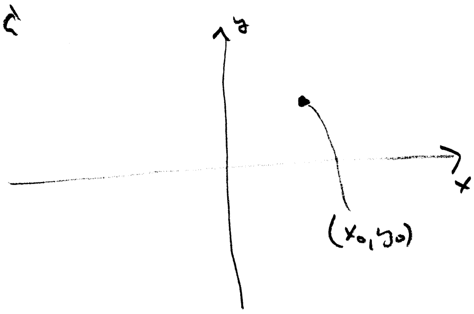
e $\quad y' = g(y) \quad$ (no explicit dependence on x)

Formally $\quad \dfrac{dy}{dx} = g(y) \quad \Leftrightarrow \quad \dfrac{dy}{g(y)} = dx$

$$\Rightarrow \quad \int \frac{dy}{g(y)} = \int dx = x + c$$

With initial conditions:

$$\int_{y_0}^{y} \frac{d\eta}{g(\eta)} = \int_{x_0}^{x} dx = x - x_0$$



Let $\quad G(y) = \displaystyle\int_{y_0}^{y} \frac{d\eta}{g(\eta)}$. Then

$(*)$ $\quad \boxed{G(y(x)) = x - x_0}$, and to find

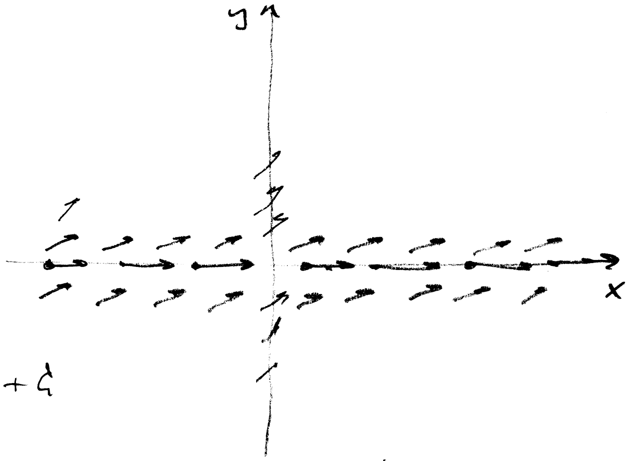
$y(x)$ we need to invert G.

But in $(*)$ there are no derivatives, so

one has <u>integrated</u> the differential equation.

It is <u>solved</u>.

Ex $\quad y' = \sqrt{|y|}$



One solution:

$y(x) = 0$

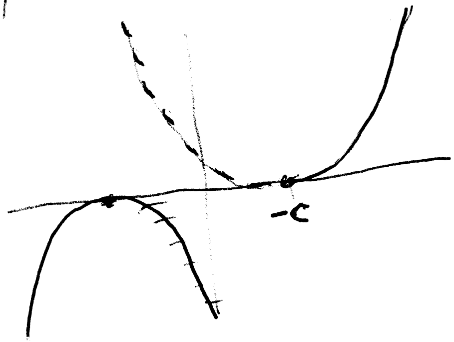
If $y > 0$

$$\frac{y'}{\sqrt{y}} = 1 \Rightarrow \int \frac{dy}{\sqrt{y}} = x + c'$$

$$\Leftrightarrow 2\sqrt{y} = x + c'$$

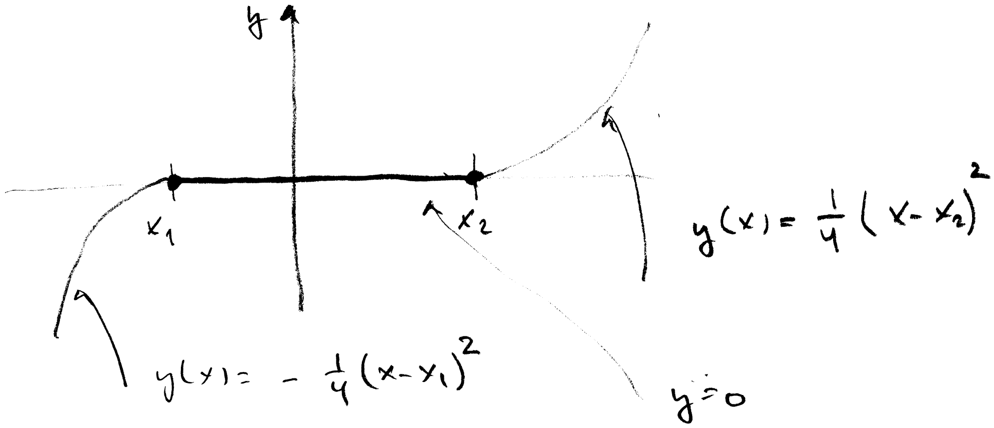
$$\Rightarrow \boxed{y = \frac{(x+c')^2}{4}} \quad (*)$$

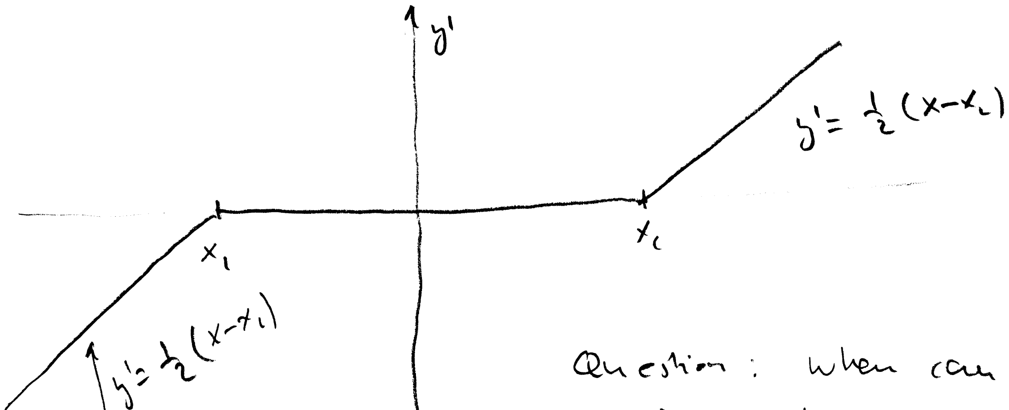
If $y < 0$: $\quad \boxed{y = -\frac{(x+c')^2}{4}}$



But note: in $(*)$ $\quad y'(x) = \frac{1}{2}(x+c') = 0$ if $x = -c'$

So $\begin{cases} y'(x) = \sqrt{|y(x)|} \\ y(0) = 0 \end{cases}$ has many solutions





Question: when can we expect a unique solution, and when can there be many?

Ex $\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$

$$\frac{dy}{y^2} = dx$$

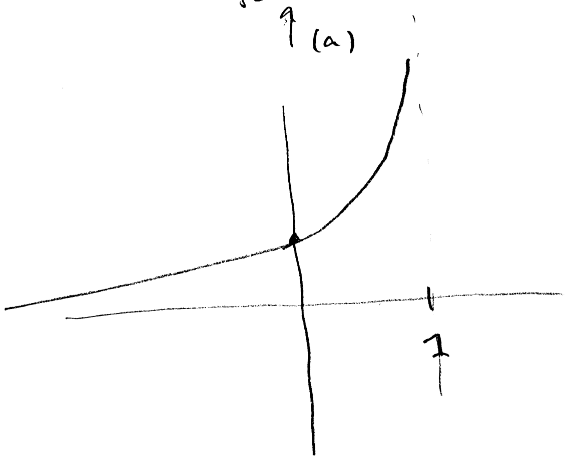
$$\Downarrow$$

$$-\frac{1}{y} = x + C_1$$

with $y(0) = 1$: $\quad -\frac{1}{1} = 0 + C_1 \Rightarrow C_1 = -1$

so the solution is

$$y(x) = \frac{1}{1-x}$$



The solution <u>explodes</u> at $x = 1$.

<u>Question:</u> Can we a-priori say anything about the interval of existence?

<u>Equations with separated variables</u>

$$\boxed{\frac{dy}{dx} = f(x)\, g(y) \qquad y(x_0) = y_0} \quad (*)$$

Heuristically:

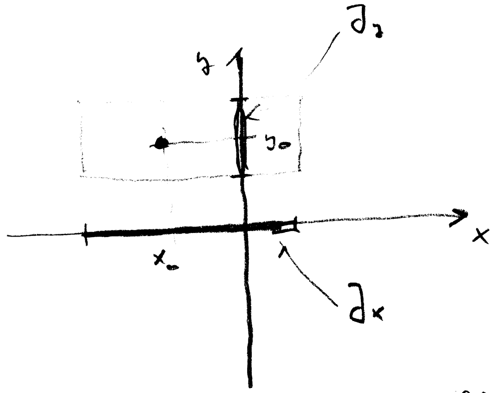
$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx \quad (+C)$$

Let $\quad G(y) = \int_{y_0}^{y} \frac{1}{g(\eta)}\,d\eta \qquad F(x) = \int_{x_0}^{x} f(\xi)\,d\xi$

and let $\quad H = G^{-1}$, i.e. $\quad H(G(y)) = y$

Then $\quad y(x) = H(F(x))$.



<u>Theorem:</u> Assume that $f(x)$ is continuous in J_x
$g(y)$ is cont. in J_y

Assume that $y_0 \in J_y$ and that $g(y_0) \neq 0$
$x_0 \in J_x$

Then the problem ⊛ has a solution that can be obtained by solving $G(y) = F(x)$ for y.

<u>Proof</u> $g(y)$ is cont. so $g(y) \neq 0$ in a neighborhood of y_0. Hence
$G(y) = \int_{y_0}^{y} \frac{1}{g(z)}\,dz$ exists near y_0, and
$G'(y) = \frac{1}{g(y)} \neq 0$ in a neighborhood of y_0.

Hence $G(y)$ has an inverse $H(z)$ defined in a neighborhood of $z_0 = G(y_0)$. Also
$H'(z) = \frac{1}{G'(y)} = g(y)$. Hence from

$$y(x) = H(F(x))$$
$$y'(x) = H'(F(x))\,F'(x) = \frac{1}{G'(y)}\,f(x) = g(y)\,f(x)$$

and $y(x_0) = H(F(x_0)) = H(0) = y_0$.