Autonomous systems phase portraits

$$(*)\begin{cases} y' = f(y) \\ y(t_0) = y_0 \end{cases} \qquad y(t) \in \mathbb{R}^n \qquad \begin{matrix} t \in J \\ (t_0 \in J) \end{matrix}$$

Def Phase space $= \mathbb{R}^n$ (or $G \subset \mathbb{R}^n$)

(in physics, the phase space often $\mathbb{R}^{2n} = \mathbb{R}^{n+n}$, where (x, v), x position, v velocity)

Def An orbit is a curve $y(J) = \{y(t) : t \in J\}$ where $y(t)$ solves $(*)$. orbit = trajectory

Def The collection of all trajectories are called the phase portrait

Note that if $y' = f(y)$ and $z' = \lambda f(z)$ then y and z follow the same orbits, only with different speed.

Def if $y(t_0) = y(t_1)$, $t_0 \neq t_1$, then $y(t)$ is periodic with period $p = |t_1 - t_0|$

The smallest p with this property is called the period of $y(t)$ (minimal period)

Def If $a \in G$ satisfy $f(a) = 0$, then a is a critical point ("stationary point", "equilibrium point")

Proposition If $y(t)$ exists for all $t \geq t_0$, and

$a = \lim_{t \to \infty} y(t)$ exists, $a \in G$, then

$f(a) = 0$.

proof Assume $\phi(t)$ is realvalued, C^1 and

$\lim_{t \to \infty} \phi'(t) = \alpha \neq 0$. If $\alpha > 0$, $\phi'(t) > \frac{\alpha}{2}$ for t large.

Then $\lim_{t \to \infty} \phi(t) = \infty$. By hypothesis $\lim y'(t) = f(a)$

$\Rightarrow \lim_{t \to \infty} y_k'(t) = f_k(a)$, but $|y_k(t)|$ - bounded $\Rightarrow f_k(a) = 0$.

Example the pendulum.

$x'' + \sin x = 0$,

Let $y_1 = x$, $y_2 = x'$ $\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = -\sin y_1 \end{cases}$

Here the phase portrait can be constructed via the Lyapunov function

$$H(y_1, y_2) = \frac{1}{2} y_2^2 - \cos y_1$$

becau

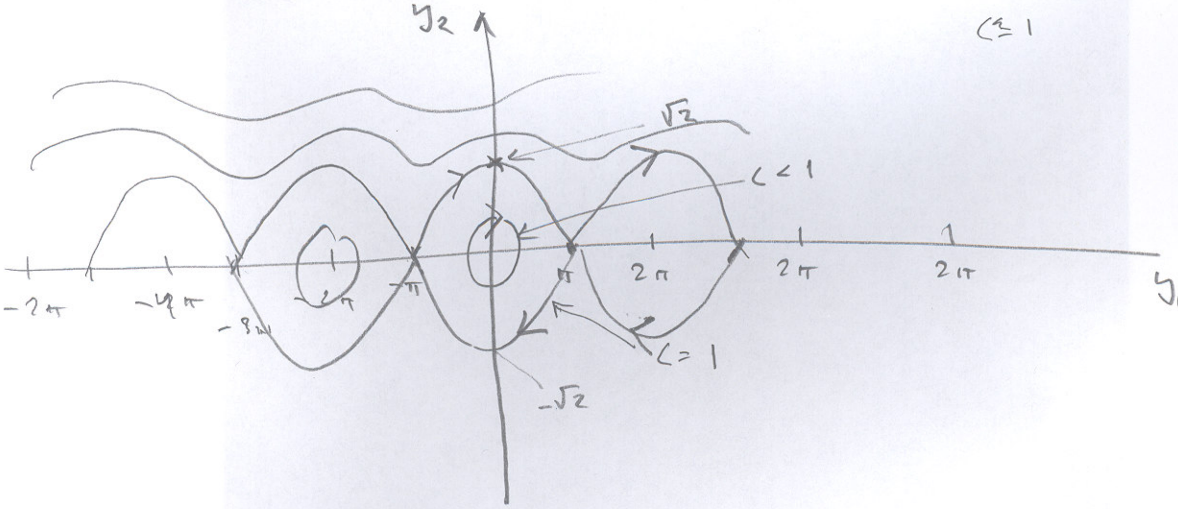
$$\frac{d}{dt} H(y_1, y_2) = \frac{\partial H}{\partial y_1} \frac{dy_1}{dt} + \frac{\partial H}{\partial y_2} \frac{dy_2}{dt}$$

$$= \sin y_1 \frac{dy_1}{dt} + y_2 \frac{dy_2}{dt} = y_2 \sin y_1 - y_2 \sin y_1 = 0$$

$\Rightarrow H(y_1, y_2)$ is const on all orbits and

$$H(y_1, y_2) = C \iff \frac{1}{2} y_2^2 - \cos y_1 = C'$$

$$\iff \quad y_2 = \pm \sqrt{C' + \cos y_1}$$



$y_2 > 0 \Rightarrow y_1' > 0$

The stationary points are $\begin{cases} y_2 = 0 \\ y_1 = k\pi \end{cases} \quad k = -\dots -3, -2, -1, 0, 1, \dots$

The curves corresponding to $C = 1$ are called <u>separatrices</u>.

<u>Example</u> $\quad x'' + \varepsilon x' + \sin x = 0$

$$\iff \begin{cases} y_1' = y_2 \\ y_2' = -\sin y_1 - \varepsilon y_2 \end{cases}$$

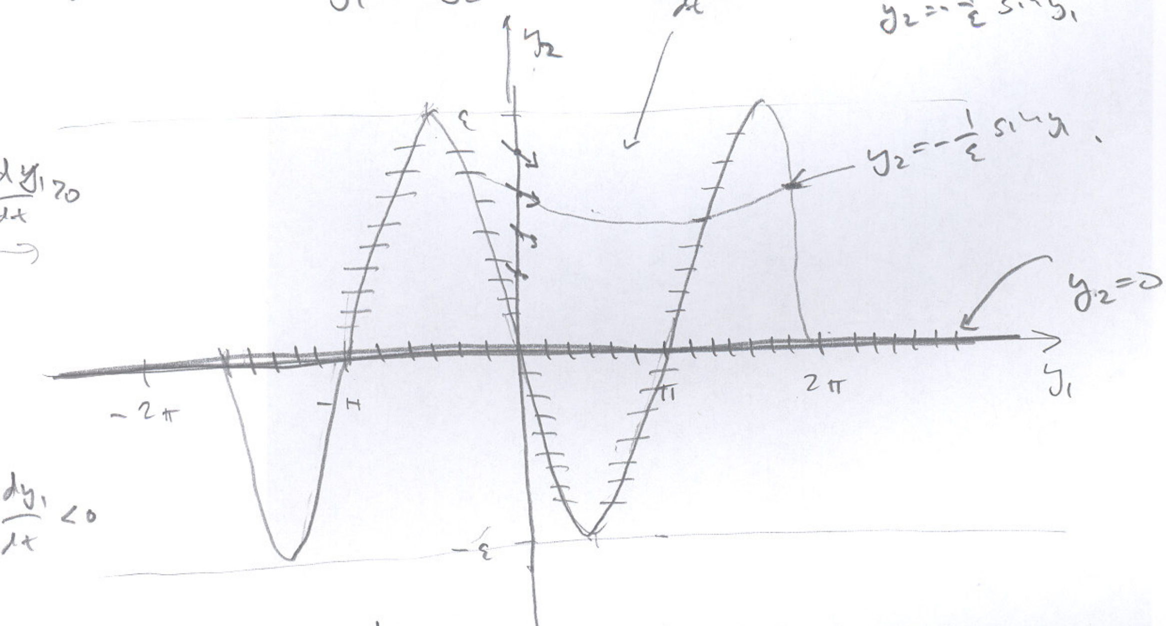
<u>Def</u> If $y' = f(y)$, $\quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad f(y) = \begin{pmatrix} f_1(y) \\ \vdots \\ f_n(y) \end{pmatrix}$

then the set of solutions to $f_k(y) = 0$ is called a <u>null cline</u>.

A null cline is typically an $n-1$ -dimensional (hyper) surface in $\mathbb{R}^n$.

In the example, the nullclines are $y_2=0$

and $-\sin y_1 - \varepsilon y_2 = 0$

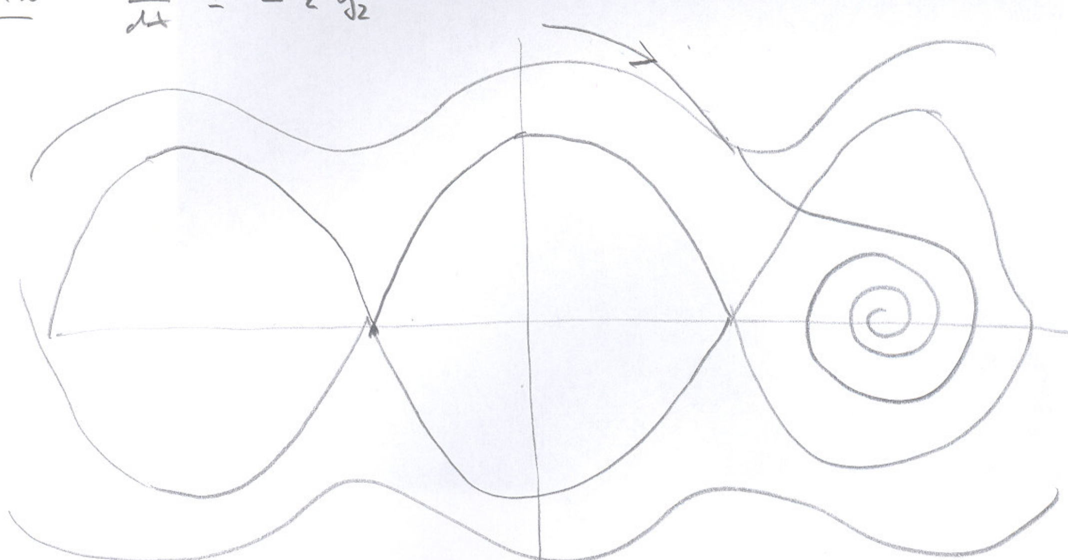


On $y_2=0$, $\frac{dy}{dt}$ is vertical, because $\frac{dy_1}{dt} = 0$.

On $y_2 = -\frac{1}{\varepsilon}\sin y_1$, $\frac{dy}{dt}$ is horizontal because $\frac{dy_2}{dt} = 0$.

On $x_1 = k\pi$, $y_2' = -\varepsilon y_2 \Rightarrow \frac{dy}{dt}$ points towards $\{y_2=0\}$

<u>Also</u> $\frac{dH}{dt} = -\varepsilon y_2^2$



An orbit to ~~∞~~ will cross the levelsets of H in a given direction

Def A function $h(y) : \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$\frac{d}{dt} h(y(t)) = 0$$

for any solution to $y' = f(y)$

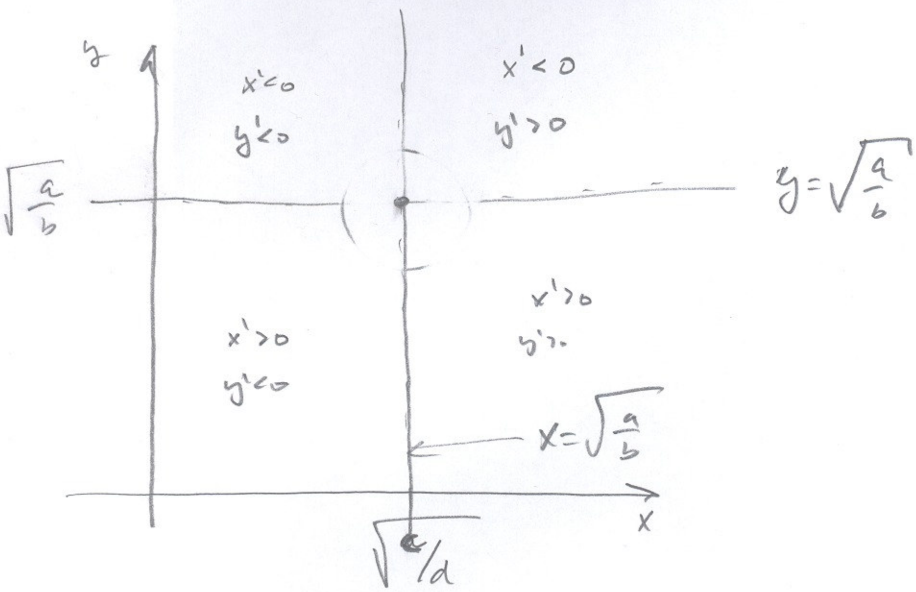
is called a <u>first integral</u>

Ex Consider positive solutions of

$$(*) \begin{cases} x' = x(a - by^2) \\ y' = y(-c + dx^2) \end{cases} \qquad \text{(a kind of Lotka Volterra model)}$$

Null clines: $x = 0$ or $y^2 = \frac{a}{b}$

$y = 0$ or $x^2 = \frac{c}{d}$



Do periodic solutions exist?

Consider: $\frac{x'}{x} = a - by^2$

$$\frac{y'}{y} = -c + dx^2$$

Let $\xi = \log x \quad \eta = \log y$

Theorem Consider

$$(*) \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

Let C be a closed (Jordan) curve and assume $(f,g) \neq (0,0)$ on C.

Let $(x(t), y(t))$ be a solution to $(*)$ with a maximal interval $t \in J^0 = (a,b)$ (open).

If $(x(J_0), y(J_0)) \subset C$, then $(x(J_0), y(J_0)) = C$ and $J^0 = \mathbb{R}$.

$$\Rightarrow \begin{cases} \xi' = a - b\exp(2\eta) \\ \eta' = -c + d\exp(2\xi) \end{cases}$$

note that $\xi' = g_1(\eta)$
$\eta' = g_2(\xi)$

so if we take $h(\xi,\eta) = \int g_1(\eta)\,d\eta - \int g_2(\xi)\,d\xi$

then $\frac{d}{dt} h(\xi,\eta) = g_1(\eta)\frac{d\eta}{dt} + g_2(\xi)\frac{d\xi}{dt}$

$$= g_1(\eta)\,g_2(\xi) - g_2(\xi)\,g_1(\eta) = 0$$

$\Rightarrow$ $h(\xi,\eta)$ is a first integral.

In our case:

$$h(\xi\eta) = a\eta - \frac{b}{2}\exp(2\eta) + c\xi - \frac{d}{2}\exp(2\xi),$$

or expressed in (x,y),

$$H(x,y) = a\log y + c\log x - \frac{b}{2}y^2 - \frac{d}{2}x^2.$$

This function has a maximum when $\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0$

$$\frac{\partial H}{\partial x} = \frac{c}{x} - dx \qquad \frac{\partial H}{\partial y} = \frac{a}{y} - by \quad \Rightarrow (x,y) = \left(\sqrt{\frac{c}{d}}, \sqrt{\frac{a}{b}}\right)$$

So the level curves are closed curves.

This implies that ⊛ has periodic solution.