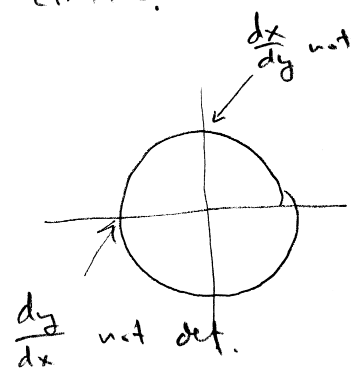
Equations for families of curves

Ex $\quad x^2+y^2=r^2 \quad (r>0)$ $\quad$ concentric circles.



satisfies $\quad 2x + 2y\frac{dy}{dx} = 0$

i.e. $\quad \frac{dy}{dx} + \frac{x}{y} = 0 \qquad (y \neq 0)$

or $\quad 2x\frac{dx}{dy} + 2y = 0$

i.e. $\quad \frac{dx}{dy} + \frac{y}{x} = 0$

A symmetric form: $\quad x\,dx + y\,dy = 0$

Interpretation: make a parameterization of the curve: $\quad (x(s), y(s)) \quad , \quad s \in I \subset \mathbb{R}$

(e.g. $\quad (x(s), y(s)) = (r\cos s, r\sin s)$)

$$\frac{d}{ds}\cdot(x(s), y(s)) = (\dot{x}(s), \dot{y}(s))$$

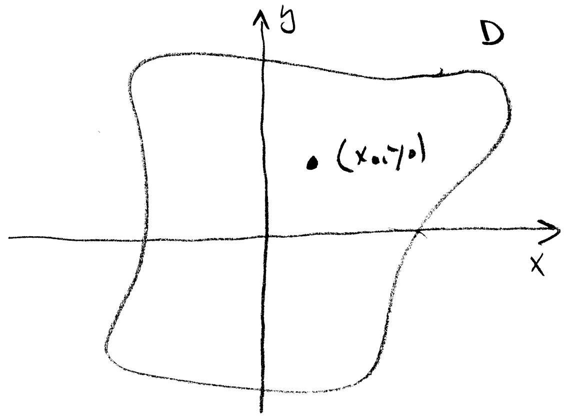
Existence and uniqueness of solutions to the initial value problem

(Ch. II in Walter, Ch I in Andersson - Böiers)

<u>Note</u> the proof is written explicitly for a SCALAR equation, but it is almost word by word the same for a system.

<u>Problem</u> Consider the equation

$$\begin{cases} y'(x) = f(x,y) & (x,y) \in D \subset \mathbb{R}^2 \\ y(x_0) = y_0 & (x_0,y_0) \in D \end{cases} \qquad f \in C(D)$$



<u>Def</u> Let $f(x,y)$ be continuous for all $(x,y) \in D$. f is said to satisfy a Lipschitz condition (is Lipschitz continuous with respect to y) if there is a constant L such that

$$|f(x,y) - f(x,z)| \le L\,|y-z|$$

holds for all $(x,y) \in D$ and $(x,z) \in D$.

<u>Note</u> the definition is exactly the same for $y, z \in \mathbb{R}^n$

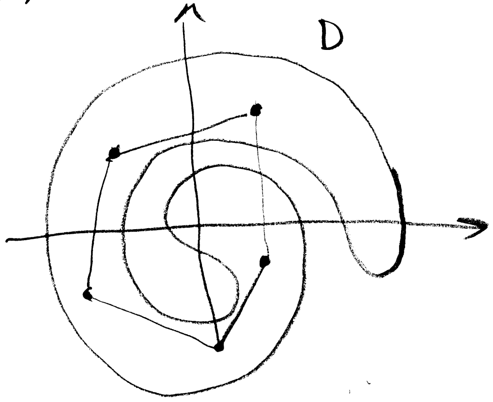
Ex If $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq K$ for some constant K, then f satisfies a Lipschitz condition, because if D is convex, then

$$f(x,y) - f(x,z) = \int_y^z \frac{\partial f}{\partial y}(x,s)\, ds = (z-y)\, \frac{\partial f}{\partial y}(x,\bar{s})$$

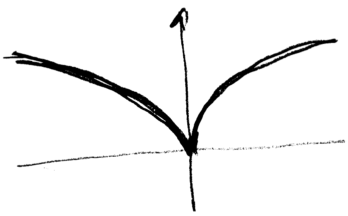
for some $\bar{s} \in \,]y,z[$ (assuming $z > y$).

Hence $|f(x,y) - f(x,z)| = |z-y| \left|\frac{\partial f}{\partial y}(x,\bar{s})\right| \leq K\,|z-y|$.

If D is not convex one has to change the argument (see fig) e.g. by constructing a polygone, but this is not a very realistic situation.



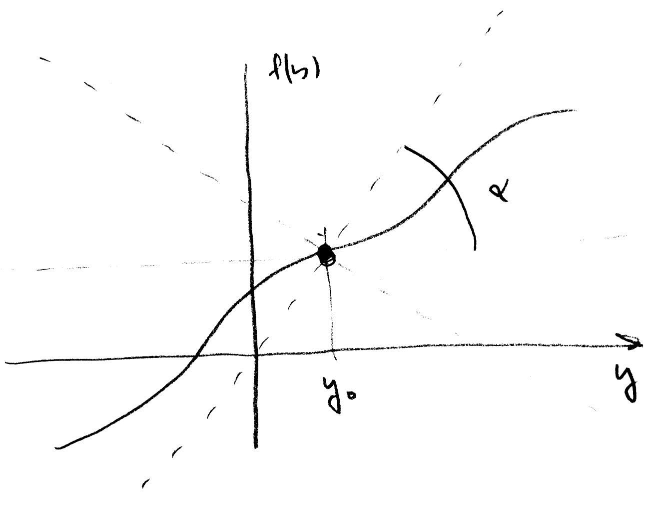
Ex $f(x,y) = \sqrt{|y|}$ is not Lipschitz continuous:



$$|f(x,0) - f(x,y)| = \sqrt{y} = \underbrace{\frac{\sqrt{y}}{|y-0|}}\,|y-0|$$

$\uparrow \to \infty$ when $y \to 0$.

Picture of Lipschitz condition:



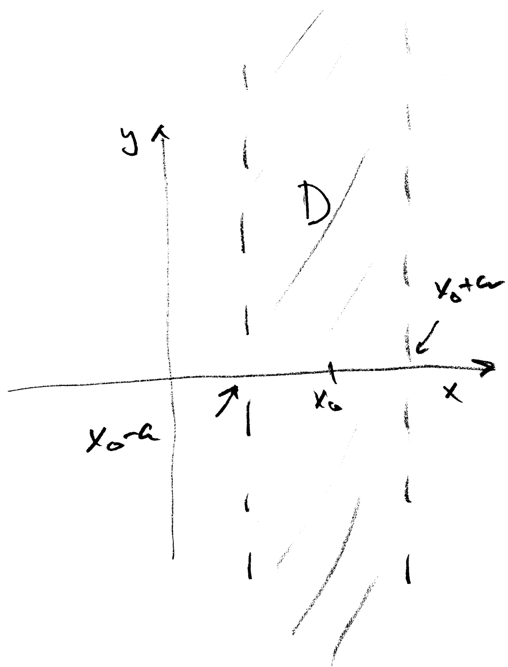
For a fixed x define the curve $(y, f(x,y))$ as in the picture. There is a cone with angle α ($\tan\alpha = L$) and center at $(y_0, f(y_0))$ such that the whole curve is inside the cone.

α should not depend on x.

Theorem 1

Assume that $f(x,y)$ is cont. and satisfies a Lipschitz condition in

$$D = \{(x,y) : |x-x_0| < a\}$$



Then the initial value problem

$$(*) \quad \begin{cases} y' = f(x,y) & (x,y) \in D \\ y(x_0) = y_0 \end{cases}$$

has a unique solution for $|x-x_0| < \varepsilon$

Proof From $(*)$ we obtain

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

we define

$$Ty = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

Then $T: C^0 \to C^0$, i.e. if f is continuous, then so is $Ty(x)$.

A solution to $(*)$ then satisfies $y = Ty$.

So $(*)$ has been transformed to the fixed point problem

$$(**) \qquad y = Ty$$

Suppose that $y(x)$ is cont and that $y = Ty$. Then $y(x)$ is differentiable, because

y cont. $\Rightarrow$ $f(x, y(x))$ cont. $\Rightarrow$

$$Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dx$$

is differentiable.

Also $Ty(x_0) = y_0 \Rightarrow y(x_0) = y_0$

and $\frac{d}{dx}\left(y_0 + \int_{x_0}^{x} f(t, y(t))dt \right) = f(x, y(x))$

So $y(x)$ satisfies $y' = f(x,y)$.

Therefore $(*)$ and $(**)$ are equivalent.

Define a seq. of functions $y_0(x), y_1(x), \dots y_n(x) \dots$

where $y_0(x) = y_0$ (a constant function)

$$y_{n+1}(x) = Ty_n(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\,dt$$

To prove existence of a solution to $(*)$ we prove that there is $y(x)$ such that ①

$y_n(x) \to y(x)$ uniformly in $|x - x_0| < a$

and ② such that $f(x, y_n(x)) \to f(x, y(x))$

uniformly in $|x - x_0| < a$.

If ① holds, then ② holds, because for all x,

$$|f(x, y_n(x)) - f(x, y(x))| \le L\,|y_n(x) - y(x)|$$

Then $\sup_x |f(x, y_n(x)) - f(x, y(x))| \le L \sup_x |y_n(x) - y(x)|$

$\to 0$ when $n \to \infty$.

From this we may also conclude that

$$\int_{x_0}^{x} f(t, y_n(t))\,dt \to \int_{x_0}^{x} f(t, y(t))\,dt$$

when $n \to \infty$.

Consider now the sequence

$$y_{n+1}(x) = y_0 + \sum_{k=0}^{n} \left(y_{k+1}(x) - y_k(x) \right).$$

We wish to prove that the series

$$\sum_{k=0}^{\infty} \left(y_{k+1}(x) - y_k(x) \right) \quad \text{is uniformly convergent}$$

in $|x - x_0| < a$.

Recall: A series $\sum_{k=0}^{\infty} f_k(x)$ is uniformly for $x \in I$ convergent if there is a function $f(x)$ such that for any $\varepsilon > 0$ there is $N > 0$ so that for all $n > N$,

$$\sup_{x \in I} \left| f(x) - \sum_{k=0}^{n} f_k(x) \right| < \varepsilon .$$

Let $K = \sup_{|x - x_0| < a} |f(x, y_0)|$.

Then $$|y_1(x) - y_0(x)| = \left| \int_{x_0}^{x} f(t, y_0(t))\, dt \right| \leq K\, |x - x_0|$$

Next (here we assume $x > x_0$, ... it is easy to modify for $x < x_0$)

$$|y_2(x) - y_1(x)| = \left| \int_{x_0}^{x} \left(f(t, y_1(t)) - f(t, y_0(t)) \right) dt \right|$$

$$\leq \int_{x_0}^{x} | f(t, y_1(t)) - f(t, y_0(t)) |\, dt$$

$$\leq L \int_{x_0}^{x} |y_1(t) - y_0(t)|\, dt \leq KL \int_{x_0}^{x} |t - x_0|\, dt = KL \frac{(x - x_0)^2}{2}$$

By induction (please do in detail!!!)

$$|y_{k+1}(x) - y_k(x)| \leq \frac{KL^k}{(k+1)!} (x - x_0)^{k+1}$$

and if $x < x_0$,

$$|y_{k+1}(x) - y_k(x)| \leq \frac{KL^k}{(k+1)!} |x - x_0|^{k+1}$$

It follows that the series $\sum_{k=0}^{\infty} (y_{k+1}(x) - y_k(x))$ is absolutely convergent, because

$$\sum_{k=0}^{\infty} \frac{K L^k}{(k+1)!} |x-x_0|^{k+1} = \frac{K}{L} e^{L|x-x_0|}.$$

Hence the sequence $\{y_n(x)\}$ is uniformly convergent, $y_n(x) \to y(x) = y_0 + \sum_{k=0}^{\infty} (y_{k+1}(x) - y_k(x))$

We also note that each $y_n(x)$ is continuous and that the uniform limit of a sequence of continuous functions is also continuous, so $y(x)$ is continuous.

The function $y(x)$ is a solution to the initial value problem. It remains to prove that there is no other solution.

Suppose then that $y(x)$ and $z(x)$ are two solutions:

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\,dx$$

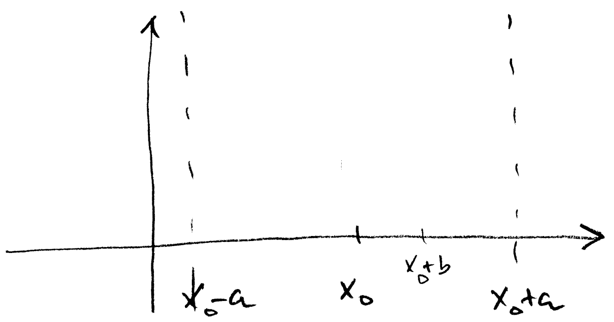
$$z(x) = y_0 + \int_{x_0}^{x} f(x, z(x))\,dx$$

Then

$$|y(x) - z(x)| = \left| \int_{x_0}^{x} (f(t, y(t)) - f(t, z(t)))\,dt \right|$$

$$\leq \int_{x_0}^{x} L\,|y(t) - z(t)|\,dt \leq L|x-x_0| \sup_{x_0 \leq t \leq x} |y(t) - z(t)|$$

if $x > x_0$, otherwise $\int_x^{x_0}$



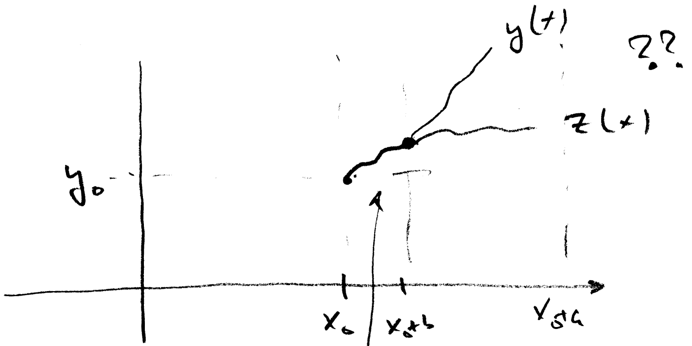
For all $x \in [x_0, x_0+b]$ we have

$$\sup_{x_0 \le x \le x_0+b} |y(x)-z(x)| \le b \cdot L \sup_{x_0 \le t \le x_0+b} |y(t)-z(t)|$$

and if $bL < 1/2$ then

$$(1-bL) \sup_{x_0 \le x \le x_0+b} |y(x)-z(x)| \le 0 \quad \Rightarrow \quad y(x) = z(x)$$

for all $x \in [x_0, x_0+b]$.



here $y(x)$ and $z(x)$ must be equal. Can they split at x_0+b?

They cannot, because they are continuous, and for $x \in [x_0+b, x_0+2b]$

$$|y(x)-z(x)| = \left| \int_{x_0+b}^{x} (f(t,y(t)) - f(t,z(t)))\,dt \right|$$

$$\le bL \sup_{x_0+b \le x \le x_0+2b} |y(t)-z(t)|$$

as before and so $y(x) = z(x)$ also for $x_0+b \le x \le x_0+2b$.

This can be iterated to cover the full interval

$x_0-a < x < x_0+a$.