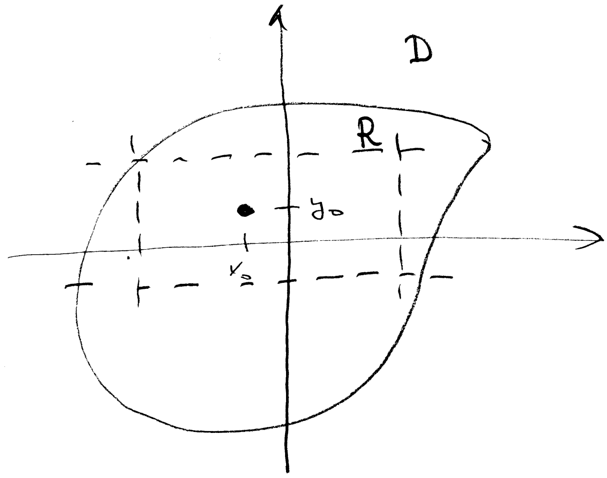
Theorem 2

Suppose $f(x,y) \in C^0(D)$, $D \subset \mathbb{R}^2$, and that f satisfies a Lipschitz condition in D. Let $(x_0, y_0) \in D$.

Then the IVP

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

has a solution $y(x)$, defined on $|x_0 - x| < \alpha$.



Proof. The difference is that the Lipschitz condition for f doesn't hold for all $y \in \mathbb{R}$.

Because $(x_0, y_0) \in D$ and D is open there is a rectangle, $R \subset D$

$$R = \{ (x,y) \in D : |x - x_0| \le a_0, \ |y - y_0| \le b \}$$

In the construction of a solution for Theorem 1, we see that

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(\xi, y_n(\xi))\, d\xi \quad , \text{ so that}$$

$$|y_{n+1}(x) - y_0| = \left| \int_{x_0}^{x} f(\xi, y_n(\xi))\, d\xi \right| \le |x - x_0| \max_{(\xi,\eta) \in R} |f(\xi,\eta)|$$

$$\le |x - x_0| M \quad (*)$$

If we take the number a in Theorem 1 as

$$a = \min\left(a_0, \frac{b}{M} \right),$$

then all functions $y_n(x)$ of the proof remain in the rectangle R, and the proof can be carried out as in Theorem 1

- -

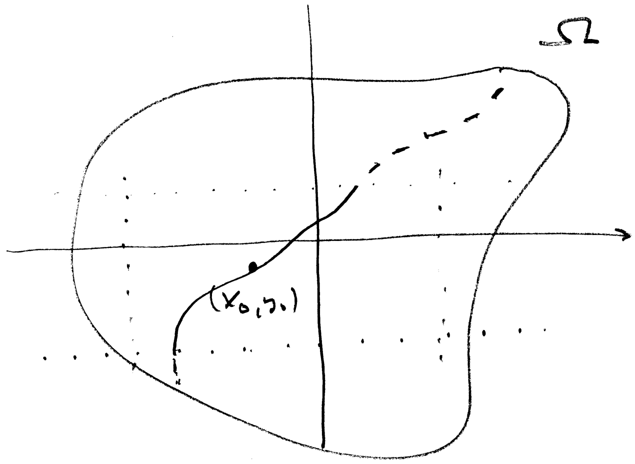
$(*)$ $M = \max_{(\xi,\eta) \in R} |f(\xi,\eta)|$

Maximal solutions

Def If $f_1 : I_1 \to \mathbb{R}$ where $I_1 \subset \mathbb{R}$
and $f_2 : I_2 \to \mathbb{R}$ $I_2 \subset \mathbb{R}$,
and $I_1 \subset I_2$, $f_1(x) = f_2(x)$ for all $x \in I_1$,
then f_2 is an <u>extension</u> of f_1

Consider

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$



where $f \in C^0(\Omega)$ satisfies a Lipschitz condition.

By theorem 2 there is a solution (unique) $y(x)$ defined on an interval $|x - x_0| < a$.

Can this be extended? If not it is called a <u>maximal solution</u>, i.e.

Let $x_M = \sup \{ \xi :$ the solution can be extended to $[x_0, \xi[\}$

and $x_m = \inf \{ \xi :$ the solution can be extended to $]\xi, x_0] \}$.

A solution defined on $]x_m, x_M[$ is <u>maximal</u>

Theorem

Assume that $f \in C^0(D)$, $D \subset \mathbb{R}^2$ (open) and that f satisfies a Lipschitz condition in a neighborhood of each point $(x,y) \in D$.

Let K be an arbitrary closed and bounded set $K \subset D$ (so K is compact).

Let $y(x)$ be a maximal solution to

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}, \qquad x \in \,]x_m, x_M[$$

where x_m and x_M are defined as before.

I if $|x_M - x|$ is sufficiently small then $(x, y(x)) \in D \setminus K$

II if $|x_m - x|$ is suff. small, then $(x, y(x)) \in D \setminus K$.

Alternative formulation: if $y(x)$ is a maximal solution then either the curve $(x, y(x))$ approaches infinity (so $x_M = \infty$ or $y(x) \to \infty$ when $x \to x_M$) or the curve comes arbitrarily close the the boundary ∂D; the same holds in the left end of $(x, y(x))$.

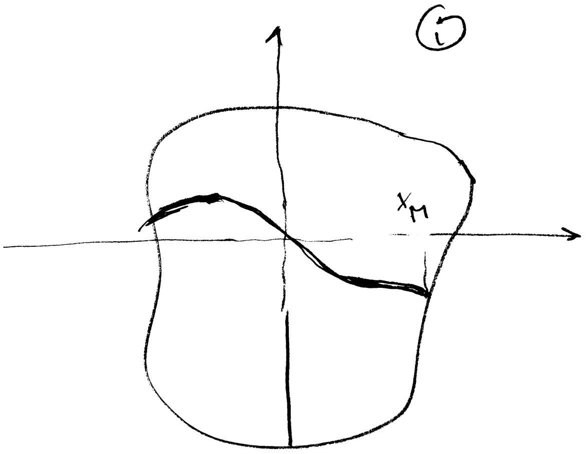
There are three possibilities:

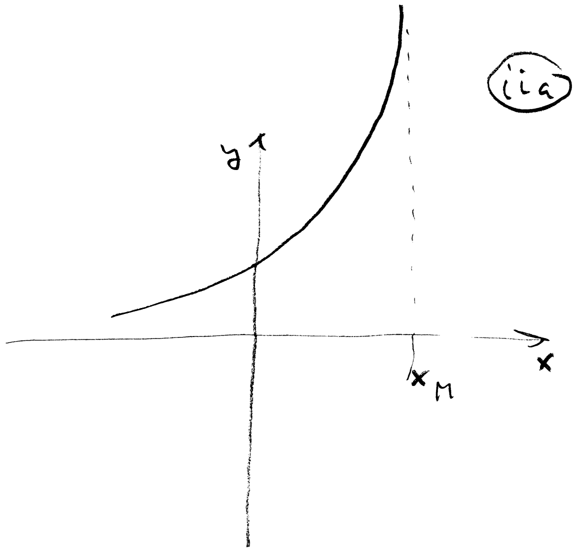
i) $(x, y(x)) \to (\bar{x}, \bar{y}) \in \partial D$ when $x \to x_M$ (or x_m)

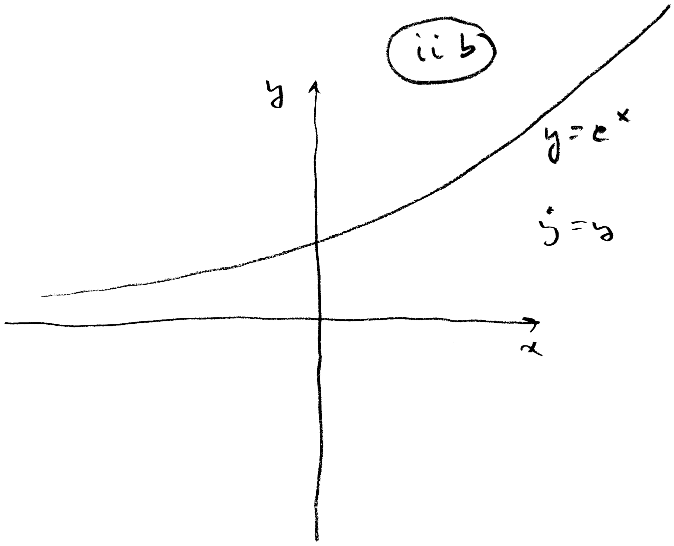
ii) $|x| + |y(x)| \to \infty$ when $x \to x_M$ (or x_m)

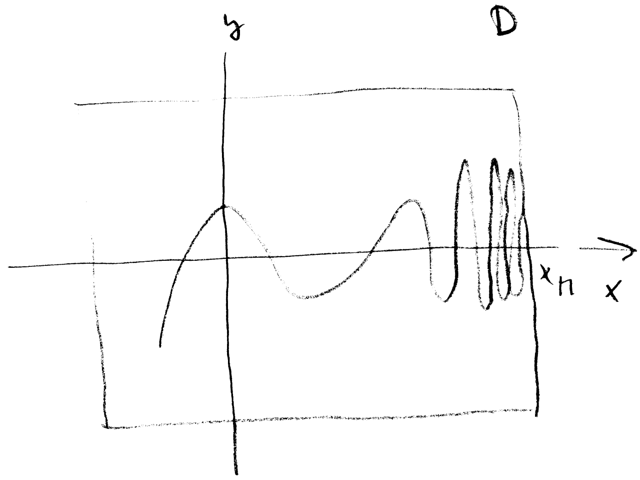
iii) $\operatorname{dist}\big((x, y(x)), \partial D\big) \to 0$ when $x \to x_M$ (or x_m) but $\lim_{x \to x_M} (x, y(x))$ does not exist.

Images :









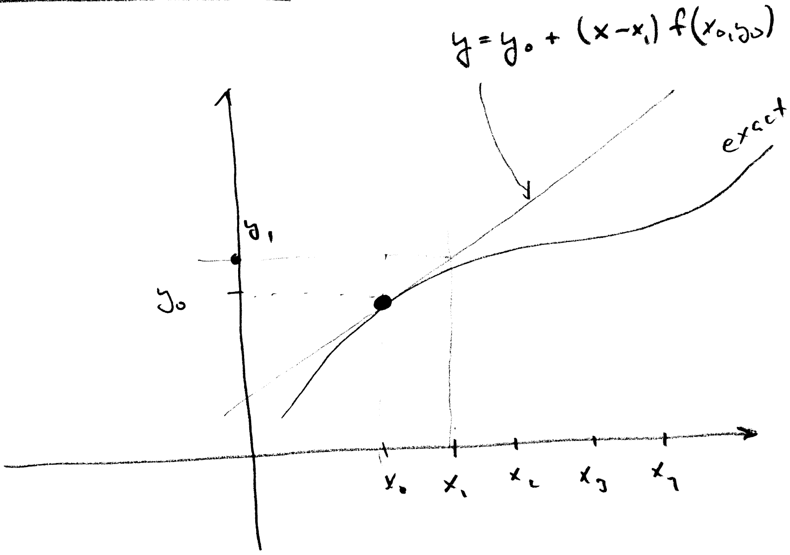
Proof Assume the contrary to the theorem.
Then there is a compact set $K \subset D$ and
a sequence $\{x_j\}$, $x_j \to x_M$ when $j \to \infty$
and such that $(x_j, y(x_j)) \in K$ $j = 1, \dots, \infty$

We must then have $x_M < \infty$ (K is bounded)
and because K is compact, there is a
subsequence x_{j_k} so that $(x_{j_k}, y(x_{j_k})) \to (\bar{x}, \bar{y}) \in K$
when $k \to \infty$. But then $\bar{x} = x_M$ and
$(x_M, y(x_M))$ is a point on the solution curve.
But $(\bar{x}, \bar{y}) \in D$ and so there is an interval $|x - \bar{x}| < \alpha$
so that the solution can be extended to $[\bar{x}, \bar{x} + \alpha[$
a contradiction.

Approximative methods, numerics

Wish to solve

$$\begin{cases} y'(x) = f(x,y) \\ y(x_0) = y_0 \end{cases}$$



The Euler method

Let $\quad x_k = x_{k-1} + \delta$

Then

$$y_1 = y_0 + \delta f(x_0, y_0)$$

$$\vdots$$

$$y_{k+1} = y_k + \delta f(x_k, y_k)$$

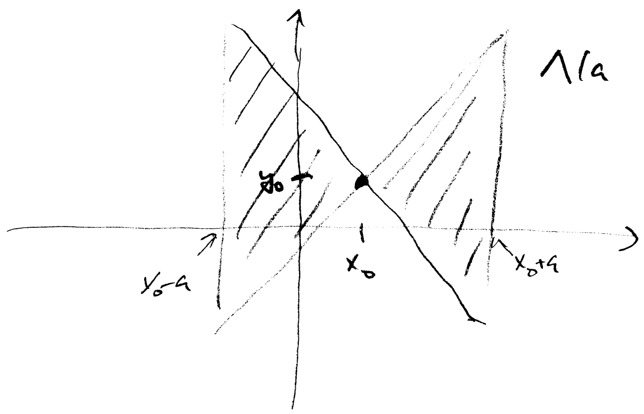
$$\vdots$$

This is called the <u>explicit</u> Euler method, because you always compute the next step explicitly.

<u>Theorem</u> Suppose $f(x,y)$ is cont in $D \subset \mathbb{R}^2$ and that $|f(x,y)| \le B$ in D and that

$$\Lambda(a) \equiv \{ (x,y) : |x-x_0| < a, \quad |y-y_0| \le B|x-x_0| \}$$

satisfies $\quad \Lambda(a) \subseteq D$.

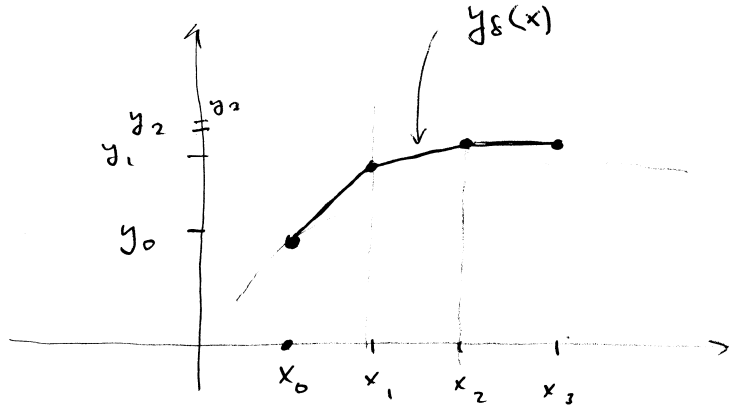


Assume also that

$$|f(x,y) - f(x,z)| \le L\,|y-z| \quad \text{and}$$
$$|f(x_1,y) - f(x_2,y)| \le C\,|x_1-x_2| \quad \text{in } D$$

Let

$$y_\delta(x) = \begin{cases} y_k & \text{when } x = x_k \\ \text{a straight line in between.} \end{cases}$$



Then the continuous function $y_\delta(x)$ satisfies

$$\left| y_\delta(x_\beta) - y_\delta(x_\alpha) - \int_{x_\alpha}^{x_\beta} f(\xi, y_\delta(\xi))\, d\xi \right| \le \varepsilon\, |x_\beta - x_\alpha| \,,$$

where $\varepsilon = \delta\,(C + LB)$

<u>Note</u> The LHS is called <u>the residual</u> and it is equal to zero for the correct solution.

The theorem shows that the residual goes to zero when the step δ decreases to zero.

<u>Proof</u> All line segments have a slope $\le B$ so for all $x_0 - a < x < x_0 + a$, $(x, y_\delta(x)) \in \Lambda(a)$.

This means that for $|x' - x''| < \delta$,

$$| f(x', y_\delta(x')) - f(x'', y_\delta(x''))| \le$$

$$\le C | x' - x''| + L | y_\delta(x') - y_\delta(x'')| \le$$

$$\le C\delta + LB\delta = (C + LB)\delta$$

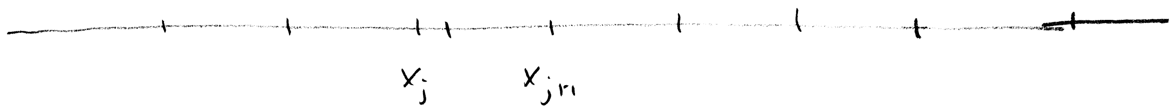
If x_β and x_α both belong to the same interval $x_j < x < x_{j+1}$, then

$$\left| y_\delta(x_\beta) - y_\delta(x_\alpha) - \int_{x_\alpha}^{x_\beta} f(\xi, y_\delta(\xi))\, d\xi \right|$$

$$= \left| (x_\beta - x_\alpha) f(x_j, y_j) - \int_{x_\alpha}^{x_\beta} f(x_j, y_j)\, d\xi \right.$$

$$\left. + \int_{x_\alpha}^{x_\beta} \left(f(x_j, y_\delta(x_j)) - f(\xi, y_\delta(\xi)) \right) d\xi \right|$$

$$\le \left| \int_{x_\alpha}^{x_\beta} \left(f(x_j, y_\delta(x_j)) - f(\xi, y_\delta(\xi)) \right) d\xi \right| \le |x_\beta - x_\alpha| \, \delta \, (C + LB)$$



This concludes the proof if x_α and x_β belong to the same interval.

If x_β and x_α belong to different intervals, one must add the errors from all intervals between x_β and x_α. The number of such intervals is $\approx \frac{|x_\beta - x_\alpha|}{\delta}$, and the error / interval is $\delta^2 (C + LB)$

$$\Rightarrow \text{residual} \le \frac{|x_\beta - x_\alpha|}{\delta} \delta^2 (C + LB) = |x_\beta - x_\alpha| \, \delta \, (C + LB).$$

Higher order methods

Heun's method

as Euler

Let $p_{k+1} = y_k + \delta f(x_k, y_k)$

$$y_{k+1} = y_k + \frac{1}{2}\delta\left(f(x_k, y_k) + f(x_{k+1}, p_{k+1})\right)$$

Runge-Kutta (Fourth order)

$k_1 = \delta f(x_n, y_n)$ (the step in Euler)

$k_2 = \delta f\left(x_n + \frac{\delta}{2}, y_n + \frac{k_1}{2}\right)$ (an "Euler-step", but computed at a different point)

$k_3 = \delta f\left(x_n + \frac{\delta}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = \delta f\left(x_n + \delta, y_n + k_3\right)$

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \qquad \left(+ O(h^5)\right)$$

How do we handle higher order equations?

Ex

$$\circledast \begin{cases} y'' + ay' + by = g \\ y(x_0) = y_{10} \qquad y'(x_0) = y_{20} \end{cases}$$

This can be transformed into a 2×2 system:

Let $y_1(x) = y(x)$

$y_2(x) = y'(x)$.

Then $\begin{cases} y_1'(x) = y'(x) = y_2(x) \\ y_1(x_0) = y_{10} \end{cases}$

$$y_2'(x) = y''(x) = g - ay' - by$$
$$= g(x) - a\, y_2(x) - b\, y_1(x)$$
$$y_2(0) = y_{20}$$

So the 2'nd order equation $\circledast$ is equivalent to the 1'st order system

$$\begin{cases} y_1' = y_2 \\ y_2' = g - a y_2 - y_1 \end{cases}$$

with the appropriate initial values.

Similarly, any equation of the form

$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ can be written

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(x, y_1, \dots, y_{n-1}) \end{pmatrix}$$

This is then in a form that can be solved by standard methods like the

Comparing solutions of differential equations

We consider differential inequalities of the form

$$y'(x) \leq f(x, y(x)) \quad , \quad f \text{ realvalued, continuous}$$

Note: here we need $y \in \mathbb{R}$, because only then $\leq$ is defined.

Lemma If $y'(x) \leq M y(x) + a$, $M \neq 0$, a const,

then $$y(x) \leq e^{M(x-x_0)} y(x_0) + \frac{a}{M}\left(e^{M(x-x_0)} - 1\right)$$

for $x \geq x_0$

Proof
$$y' - My - a \leq 0$$

$$\Leftrightarrow \quad e^{-Mx}(y' - My) - e^{-Mx} a \leq 0$$

$$\Leftrightarrow \quad \frac{d}{dx}\left(e^{-Mx} y(x) + \frac{1}{M} e^{-Mx} a \right) \leq 0$$

$$\Rightarrow \quad e^{-Mx} y(x) + \frac{a}{M} e^{-Mx} \text{ is non increasing}$$

$$\Rightarrow \quad e^{-Mx} y(x) + \frac{a}{M} e^{-Mx} \leq e^{-Mx_0} y(x_0) + \frac{a}{M} e^{-Mx_0}$$

$(x \geq x_0$

$$\Rightarrow \quad y(x) \leq e^{-M(x_0-x)} y(x_0) + \frac{a}{M}\left(e^{-M(x_0-x)} - 1\right)$$

$\blacksquare$

Lemma If $y(x) \le M \int_{x_0}^{x} y(\xi)\,d\xi + a x + b$

($M, a, b,$ constants, $M > 0$)

then

$$y(x) \le e^{M(x-x_0)}(a x_0 + b) + \frac{a}{M}\left(e^{M(x-x_0)} - 1\right)$$

$(x \ge x_0)$

Proof Let $z(x) = M \int_{x_0}^{x} y(\xi)\,d\xi + a x + b$.

Then $y(x) \le z(x)$.

And $z'(x) = M y(x) + a \le M z(x) + a$

$$\Rightarrow \quad z(x) \le e^{M(x-x_0)} z(x_0) + \frac{a}{M}\left(e^{M(x-x_0)} - 1\right)$$

(according to the previous lemma)

$$= e^{M(x-x_0)}(a x_0 + b) + \frac{a}{M}\left(e^{M(x-x_0)} - 1\right)$$

∎

This kind of estimates are often known as Gronwall type estimates (Gronwall's lemma)

Lemma If $y(x) \le M \int_{x_0}^{x} y(\xi)\,d\xi + ax + b$

(M, a, b, constants, $M > 0$)

then
$$y(x) \le e^{M(x-x_0)}(ax_0+b) + \frac{a}{M}\left(e^{M(x-x_0)} - 1\right)$$

$(x \ge x_0)$

Proof Let $z(x) = M \int_{x_0}^{x} y(\xi)\,d\xi + ax + b$.

Then $y(x) \le z(x)$.

And $z'(x) = M y(x) + a \le M z(x) + a$

$$\Rightarrow \quad z(x) \le e^{M(x-x_0)} z(x_0) + \frac{a}{M}\left(e^{M(x-x_0)} - 1\right)$$

(according to the previous lemma)

$$= e^{M(x-x_0)}(ax_0+b) + \frac{a}{M}\left(e^{M(x-x_0)} - 1\right)$$

$\blacksquare$

This kind of estimates are often known as Gronwall type estimates (Gronwall's lemma)

Comparison theorems

Consider a differential equation

$$y' = f(x,y) \qquad (y \text{ real})$$

For a continuous, differentiable function $\phi(x)$, define the residual (or defect)

$$P\phi = \phi' - f(x,\phi).$$

Theorem Let ϕ and ψ be differentiable in $J_0 = \{\xi < x < \xi + a\}$

Assume that a) $\phi(x) < \psi(x)$ for $\xi < x < \xi + \varepsilon$ $(< \xi + a)$

b) $P(\phi) < P(\psi)$ in J_0

Then $\phi < \psi$ in J_0.

Proof Assume that there is $x_0 \in]\xi+\varepsilon, \xi+a[$ such that $\phi(x_0) = \psi(x_0)$

Then
$$\begin{aligned} 0 > P(\phi) - P(\psi) &= \phi'(x_0) - f(x_0, \phi(x_0)) - (\psi'(x_0) - f(x_0, \psi(x_0))) \\ &= \phi'(x_0) - \psi'(x_0) - (f(x_0, \phi(x_0)) - f(x_0, \psi(x_0))) \\ &= \phi'(x_0) - \psi'(x_0) \end{aligned}$$

$\Rightarrow \quad \phi'(x_0) < \psi'(x_0)$

But
$$\frac{d}{dx}(\phi(x) - \psi(x))\Big|_{x=x_0} = \lim_{h\to 0} \frac{\phi(x_0) - \psi(x_0) - (\phi(x_0-h) - \psi(x_0-h))}{h}$$

$$= \lim_{h \to 0} \frac{\psi(x_0-h) - \phi(x_0-h)}{h} < 0$$

$\Rightarrow \quad \psi(x_0-h) < \phi(x_0-h)$ for some $x_0 - h$.

$\Rightarrow$ contradiction.

So $\phi < \psi$ for x in the full interval J_0.

Corollary If $\begin{cases} y' = f(x,y) \\ y(\xi) = \eta \end{cases}$ in $J = [\xi, \xi+a]$

and $\begin{cases} z' < f(x,z) \\ z(\xi) = \eta \end{cases}$, z differentiable.

Then $z(x) < y(x)$ $(x \in J)$.

Proof $z'(\xi) - y'(\xi) < 0$

and $z(\xi) = y(\xi) = \eta$

$\Rightarrow$ there is an interval $]\xi, \xi+\varepsilon[$ where $z(x) < y(x)$

Example Consider $\begin{cases} y' = x^2 + y^2 \\ y(0) = 1 \end{cases}$ $x > 0$

For $x > 0$, $x^2 + y^2 > y^2$, and for $0 \le x < 1$, $x^2 + y^2 < y^2 + 1$

$$\Rightarrow \quad y^2 < y' < y^2 + 1 \qquad (0 < x < 1)$$

And $\begin{cases} v' = v^2 + 1 \\ v(0) = 1 \end{cases}$ has solution $v = \tan\left(x + \frac{\pi}{4}\right)$

$\begin{cases} w' = w^2 \\ w(0) = 1 \end{cases}$ has solution $w = \dfrac{1}{1-x}$

$$\Rightarrow \quad \frac{1}{1-x} < y(x) < \tan\left(x + \frac{\pi}{4}\right).$$