Linear systems of differential equations

Here we consider equations of the form

$$\begin{cases} y'(x) = A(x)\, y(x) + b(x) & (x \geq x_0) \\ y(x_0) = y_0 , \end{cases}$$

where $y \in \mathbb{R}^n$, A is an $n \times n$ - matrix, that may depend on x

and $b \in \mathbb{R}^n$.

<u>Theorem</u> If $A(x)$ is continuous, $b(x)$ continuous on an interval J, and $x_0 \in J$.

Then the system

$$\begin{cases} y' = A y + b \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $y(x)$, $x \in J$.

This solution satisfies

1) if $J' \subset J$ and $|A(x)| < L$ $(x \in J')$

$|b(x)| < \delta$, $|y_0| < \eta$

then $|y(x)| \leq \eta e^{L(x-x_0)} + \frac{\delta}{L}\left(e^{L(x-x_0)} - 1\right)$ $(x \in J')$

2) for every $\varepsilon > 0$ there is $\beta > 0$ so that if

$$\begin{cases} z' = Bz + c \\ z(x_0) = z_0 \end{cases}$$

is a second linear system with

$$|B(x) - A(x)| < \beta \quad , \quad |b(x) - c(x)| < \beta \qquad (x \in J')$$

and $\quad |y_0 - z_0| < \beta$,

then $\quad |y(x) - z(x)| < \varepsilon \quad$ for $\quad x \in J'$

<u>Proof of existence and uniqueness</u>

Because $A(x)$ is continuous (i.e. all entries $a_{ij}(x)$ are continuous) we have that

$$f(x, y) = A(x)\, y + b(x)$$

is continuous and satisfies a Lipschitz condition

$$|f(x, y) - f(x, z)| = A(x)\,(y - z)$$

$$\leq n \max_{ij} \max_{x \in J} |a_{ij}(x)| \quad |y - z|$$

from which the existence and uniqueness follows.

<u>Comments</u> • Here $|y| = \sqrt{\sum y_j^2}$, the Euclidean norm

But to prove the rest of the theorem, we need to define matrix norms.

• The statements 1) and 2) of the theorem are stability results, which are very important, rather easy in the linear case but sometimes very difficult (and not always true!) for nonlinear equations.

General about linear and normed spaces

<u>Example</u> $\mathbb{R}^n$ is a linear space:

The elements (vectors), and the operation + and scaler multipl. satisfy:

1) $x, y \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{R}$ $\Rightarrow$ $\lambda x + \mu y \in \mathbb{R}^n$

2) $x + y = y + x$ 3) $(x+y)+z = x + (y+z)$

4) $\lambda(x+y) = \lambda x + \lambda y$ 5) $(\lambda+\mu)x = \lambda x + \lambda \mu$

6) $\forall\, x \in \mathbb{R}^n$ $\exists$ $(-x) \in \mathbb{R}^n$ so that $x + (-x) = 0 \in \mathbb{R}^n$

7) $\exists\ 0 \in \mathbb{R}^n$: $x + 0 = x$

8) $1 \cdot x = x$

<u>Another example:</u> $C(\mathbb{R})$ is a linear space where + is defined pointwise: $u, v \in C(\mathbb{R})$. To say tha $u + v = w$ is the same as setting

$$w(x) = u(x) + w(x)$$

<u>Anothere example</u> $V_0 = \{ v \in C(\mathbb{R}) : v(0) = 0 \}$ is a linear space, but $V_1 = \{ v \in C(\mathbb{R}) : v(0) = 1 \}$ is <u>NOT</u>

In $\mathbb{R}^n$ there is a base, $e_1 \dots e_n$ so that each $x \in \mathbb{R}^n$ can be written in a unique way as $x = \sum_{k=1}^{n} x^k e_k$. $x^1 \dots x^n$ are the coordinates.

A map $\mathbb{R}^n \to \mathbb{R}^n$ can be defined by a matrix

$$v = Au \iff \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

The matrix depends on the basis chosen.

<u>Matrices</u> (Here we consider only square matrices)

$$A + B = (a_{ij} + b_{ij}) \qquad \text{if} \qquad A = (a_{ij}) \quad B = (b_{ij})$$

$$A + B = B + A \qquad \text{but} \qquad AB \neq BA \quad \text{in general.}$$

$$C = AB \iff C = (c_{ij}) \quad \text{with}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

The determinant is defined as

$$\det A = \sum_p (-1)^{\nu(p)} a_{1p_1} a_{2p_2} \cdots a_{np_n},$$

where p is a permutation of $1, 2, \ldots, n$

<u>Norms</u> A norm is a map $V \to \mathbb{R}^+$ such that $x \mapsto |x|$

1) $|x| \geq 0$ $\quad |x| = 0 \Rightarrow x = 0$
2) For $x \in V$, $\lambda \in \mathbb{R}$, $\quad |\lambda x| = |\lambda|\,|x|$. (homogeneity)
3) $|x + y| \leq |x| + |y|$ (triangle inequality)

Norms in $\mathbb{R}^n$:

Euclidean norm

$$|y| = \sqrt{\sum_{j=1}^{n} |y_j|^2}$$

For matrixes:

$$|A|_1 = \sum_{i,j} |a_{ij}|$$

$$|A|_\infty = \max_{ij} |a_{ij}|$$

Operator norm

$$\|A\| = \sup_{|y|=1} |Ay|$$

In general, a matrix norm should satisfy

$$|AB| \leq |A|\,|B|$$

$$|Ax| \leq |A|\,|x|$$

(in addition to the usual rules)

General

A vector space V is consists of

Complete normed spaces

Let V be a vector space.

Examples: $\mathbb{R}^n$ (of course)

$C^0(\mathbb{R})$ spaces of functions

$C^1(\mathbb{R})$

$C^0_0(\mathbb{R}) = \{ x(t) \in C^0(\mathbb{R}) : |x(t)| \to 0 \quad |t| \to \infty \}$

Def A 'Cauchy sequence' is a sequence $\{x_k\}_{k=1}^{\infty}$ $x_k \in V$ such that for all $\varepsilon > 0$, there is $N_0(\varepsilon) > 0$ so that whenever $m, k > N_0(\varepsilon)$,

$$\| x_m - x_k \| < \varepsilon$$

Def A sequence $\{x_k\}_{k=1}^{\infty}$ is said to be convergent if there is $x \in V$ so that

$$\lim_{N \to \infty} \| x_N - x \| = 0$$

Def A series $\sum_{k=1}^{\infty} x_k$ is said to be convergent if the sequence $\{y_N\}$, where $y_N = \sum_{k=1}^{N} x_k$, is convergent.

Def A normed (vector) space is said to be complete if all Cauchy sequences are convergent

Ex $\mathbb{Q}$, the rational numbers, is not complete

$\{x_k\} = 3,\underbrace{1415926535}_{k \text{ decimals}}$

$x_k \to \pi \notin \mathbb{Q}$

Def A Banach space is a complete, normed linear space.

Def If E and F are normed vector spaces a mapping

$$T : E \to F$$

is called an <u>operator</u> and a mapping

$$T : E \to \mathbb{R} \quad (\text{or } \mathbb{C}) \text{ is called}$$

a real (or complex) functional

Def The operator norm of T is

$$\|T\| = \sup_{x \in V} \frac{\|Tx\|}{\|x\|} \qquad (\|x\| \neq 0)$$

Def An operator $T : E \to E$ is called a <u>contraction</u> if for all $x, y \in E$,

$$\| T(x) - T(y) \| \leq q \, \|x - y\| \qquad (q < 1)$$

The contraction mapping principle:

Every contraction has a unique fixed point, i.e. an $x_0 \in E$ such that

$$T x_0 = x_0$$

Matrices depending on a variable

$A(x) = (a_{ij}(x))$, the elements depend on the variable x.

$A'(x) = (a'_{ij}(x))$

$\int_a^b A(x)dx = \left(\int_a^b a_{ij}(x)\, dx\right)$

the derivative and integral of a matrix is defined as the derivative or integral of each element

A matrix is continuous (differentiable) if each element is continuous (differentiable).

<u>Lemma</u> $(AB)' = \frac{d}{dx}(AB) = A'B + AB'$

$(Ay)' = A'y + Ay'$

$$(\det A)' = \sum_{k=1}^{n} \det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a'_{k1} & a'_{k2} & & a'_{kn} \\ \vdots & & & \\ a_{n1} & \cdot & & a_{nn} \end{pmatrix}$$

<u>Proof</u> $C = AB \Rightarrow$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \Rightarrow c'_{ij} = \sum_{k=1}^{n} (a'_{ik} b_{kj} + a_{ik} b'_{kj})$$

$$\det A = \sum_{p} (-1)^{\nu} a_{1p_1} \cdots a_{np_n}$$

$$\frac{d}{dx}(\det A) = \sum_{p} (-1)^{\nu} \sum_{k=1}^{n} a_{1p_1} \cdots a'_{kp_k} \cdots a_{np_n}$$

Consider the linear equation

$$\circledast \quad \frac{dy}{dx} = A(x)\, y \qquad \text{(a homogeneous equation)}$$

$x \in J \subset \mathbb{R}^n \qquad y \in \mathbb{R}^n, \quad A$ is $n \times n$, continuous

A solution $y(x)$ belongs to $C^1(J, \mathbb{R}^n)$, i.e. a vectorvalued, differentiable function.

<u>Note</u> $C^1(J, \mathbb{R}^n)$ is a vectorspace.

<u>Theorem</u> Let $y_1, \dots, y_k \in C^1(J, \mathbb{R}^n)$ be solutions to $\circledast$ (note that here $y_j(x) \in \mathbb{R}^n$, the index does not refer to a coordinate of a vector) Then the following are equivalent:

1) $y_1 \dots y_k$ are linearly independent in $C^1(J, \mathbb{R}^n)$, i.e., if $\lambda_1 y_1 + \dots \lambda_k y_k = 0 \quad (\lambda_1, \dots, \lambda_k \in \mathbb{R})$ then $\lambda_1 = \lambda_2 = \dots = \lambda_k = 0$.

2) $y_1(x), y_2(x), \dots, y_k(x)$ are linearly independent in $\mathbb{R}^n$ for <u>all</u> $x \in J$

3) $y_1(x_0), y_2(x_0) \dots y_k(x_0)$ are linearly independent for <u>some</u> $x_0 \in J$.

<u>Proof</u> We shall prove that $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1)$