The step 2) $\to$ 3) is trivial.

To prove 3) $\to$ 1) suppose that

we want to prove $\lambda_1 \dots \lambda_k = 0$

$$\sum^k \lambda_i y_i = 0 \quad \text{in} \quad C^1(J, \mathbb{R}^n) \iff$$

$\sum \lambda_i y_i(x) = 0$ for all $x \in J$, in particular for any $x_0 \in J$. Then

$\sum \lambda_i y_i(x_0) = 0 \Rightarrow \lambda_i = 0, \ i = 1, \dots, k$ (because $\{y_i(x_0)\}$ are independent in $\mathbb{R}^n$. $\blacksquare$

To prove 1) $\to$ 2) Assume $\sum_{i=1}^k \lambda_i y_i(x_0) = 0$ for some $x_0 \in J$. Want to prove $\lambda_1, \dots, \lambda_k = 0$

Let $y(x) = \sum^k \lambda_i y_i(x)$. Then

$y(\cdot)$ satisfies $\begin{cases} y' = A(x) y \\ y(x_0) = 0 \end{cases} \Rightarrow y(x) \equiv 0$

$\Rightarrow y(\cdot) \equiv 0 \Rightarrow \sum \lambda_i y_i = 0$ in $C^1(J, \mathbb{R})$

$\Rightarrow \lambda_i = 0, \ i = 1, \dots, k$ because the y_i were assumed to be independent in $C^1(J, \mathbb{R})$.

<u>Theorem</u> Let $V = \{ y \in C^1(J, \mathbb{R}^n),\ y \text{ solves } y' = A(x) y \}$

Then V is an n-dimensional vector space, a subspace of $C^1(J, \mathbb{R}^n)$

<u>Proof</u> Pick any $x_0 \in J$. The set of possible initial values is n-dimensional, so one can have exactly n independent $y(x_0)$, call them $y_1(x_0), \dots, y_n(x_0)$.
The solutions to $\begin{cases} y_j' = A(x) y_j \\ y_j(x_0) = y_{j0} \end{cases}$

are then a (maximal) number of independent solutions to $y' = A(x) y$.

<u>Def</u> Let $y_1, \dots, y_n \in V$ be n independent solutions to $y' = A(x)y$.

Let
$$Y = [y_1, \dots, y_n]$$
, i.e. an $n \times n$-matrix whose columns are the n independent solutions (not that this choice is not unique!)

Then $Y(x)$ solves the equation
$$Y'(x) = A(x) Y(x)$$
, and it is called a <u>fundamental</u> solution.

<u>Some remarks</u> Sometimes it is convenient to write $y = y(x, x_0, y_0)$ for the solution to
$$\begin{cases} y' = A(x) y \\ y(x_0) = y_0 \end{cases}$$
.

Then the map $y_0 \mapsto y(x, x_0, y_0)$ is a bijective (1-1) map.

A special fundamental matrix is obtained by solving
$$\begin{cases} X' = A(x) X \\ X(x_0) = I \end{cases} \qquad \text{(all } n\times n\text{-matrices)}$$

Then ANY fundamental matrix $Y(x)$ can be obtained as
$$Y(x) = X(x) Y(x_0)$$
(exercise: prove this)

The Wronskian

Let $Y(x)$ be a solution of $Y' = A(x) Y$.

Let $\phi(x) = \det Y(x)$. $\phi(x)$ is called the Wronskian of Y.

Theorem If $A(x)$ is continuous in J, then $\phi(x)$ satisfies

$$\phi'(x) = \operatorname{tr} A(x)\, \phi(x) \quad \text{in } J.$$

(Recall that $\operatorname{tr} M = \sum_{j=1}^{n} m_{jj}$, for an $n \times n$ matrix $M = (m_{ij})$; this is called the trace of M)

Proof Let X solve $\begin{cases} X' = A(x) X \\ X(x_0) = I \end{cases}$

Then $(\det X(x))' = \sum_{i=1}^{n} \det [x_1 \; x_2 \; x_i' \; x_n]$

But $x_i'(x_0) = A(x_0) e_i$ (because $x_j(x_0) = e_j$)

and therefore

$$(\det X(x_0))' = \sum_{i=1}^{n} \det(e_1 \dots, e_{i-1}, A(x_0) e_i, e_{i+1}, \dots e_n)$$

$$= \sum_{i=1}^{n} a_{ii}(x_0) = \operatorname{tr} A(x_0).$$

But $Y(x) = X(x) Y(x_0)$

$\Rightarrow \det Y(x) = \det X(x) \det Y(x_0)$

$\Rightarrow \phi'(x) = \phi(x_0) \, (\det X(x))'$

$\Rightarrow \phi'(x_0) = \phi(x_0) \operatorname{tr} A(x_0)$ //

Note hence the Wronskian is always $\neq 0$, if that holds at one single point.

Proof of continuous dependence

We want to prove 1) and 2) on page 38.

1). $y' = Ay + b$. Note that , $|y(x_0)| < \eta$

$$\frac{d}{dt}|y|^2 = 2\, y \cdot y' = 2\, y^{tr} y'$$

and $|y^{tr} A y| \leq \|A\| \, |y|^2$ (now we consider the operator norm of A)

$$|y^{tr} b| \leq |y|\,|b|$$

Then if $|y| \neq 0$
$$\frac{d}{dt}|y|^2 \leq 2\|A\|\,|y|^2 + 2|b|\,|y|$$

$\|A\| < L$, $|b| < \delta$, $|y_0| < \eta$

$$\Rightarrow \quad 2|y| \frac{d|y|}{dt} \leq 2\|A\|\,|y|^2 + 2|b|\,|y|$$

$$\Rightarrow \quad \frac{d|y|}{dt} \leq \|A\|\,|y| + |b|$$

$$\Rightarrow \quad |y| \leq \eta\, e^{L(x-x_0)} + \frac{\delta}{L}\left(e^{L(x-x_0)} - 1\right)$$

according to the lemma page 34.

∎

2) We give a proof of this statement which is based on 1). Note that the norm used for matrices is the operator norm, but any compatible norm could have been used.

Consider y and z, solutions to

$$\begin{cases} y' = Ay + b \\ y(x_0) = y_0 \end{cases} \qquad \begin{cases} z' = Bz + c \\ z(x_0) = x_0 \end{cases}$$

Let $J' \subset J$ be a compact (i.e. closed and bounded) interval. Let $L_1 = \max_{x \in J'} \|A\|$, $L_2 = \max_{x \in J'} \|B\|$,

$\eta_1 = |y(x_0)|$ $\quad$ $\eta_2 = |z(x_0)|$

$\delta_1 = \max_{x \in J'} |b|$ $\quad$ $\delta_2 = \max_{x \in J'} |c|$

Let $w = y - z$. Then

$$w' = Ay - Bz + b - c = A(y-z) + (A-B)z + (b-c)$$

If $\gamma = |w_0| = |y(x_0) - z(x_0)|$ and $\delta = \max_{x \in J'} |(A-B)z + (b-c)|$,

then 1) implies that

$$|w(x)| \le \gamma e^{L_1|x-x_0|} + \frac{\delta}{L_1}\left(e^{L_1|x-x_0|} - 1\right) \quad \text{in } J'.$$

Let $\rho = |J'|$, the length of the interval J'.

If we can achieve $\gamma + \frac{\delta}{L'} < \varepsilon / e^{L_1\rho}$, then we are done.

But $\gamma \le \beta$. We can use 1) to estimate δ, in the following way:

$$|(A-B)z + (b-c)| \le |(A-B)z| + |b-c| \le \|A-B\|\,|z| + |b-c| \le \beta\,|z(x)| + \beta$$

according to the hypothesis for 2). But according to 1),

$$|z| \le \eta_2 e^{L_2\rho} + \frac{\delta_2}{L_2}\left(e^{L_2\rho} - 1\right).$$

Note that because $\|B-A\|$ should be small, and $|b-c|$ and $|y(x_0) - z(x_0)|$ too, it is reasonable to estimate η_2 by $2\eta_1$ etc, and hence

$$|z| \le 2\eta_1 e^{2L_1\rho} + \frac{\delta_1}{L_1}\left(e^{2L_1\rho} - 1\right) \quad \text{for all } B \text{ close to } A$$

Hence $\gamma + \frac{\delta}{L'} < \beta + \beta\left(2\eta_1 e^{2L_1\rho} + \frac{\delta_1}{L_1}\left(e^{2L_1\rho} - 1\right)\right) + \beta$

which is smaller than $\varepsilon / e^{L_1\rho}$ if β is small enough.

Inhomogeneous differential equations

We wish to solve

$$\begin{cases} y' = A(x)\, y + b(x) & y(x) \in \mathbb{R}^n \quad x \in J \\ y(x_0) = y_0 & x_0 \in J \end{cases}$$

Note If $\tilde{y}$ is any solution $y' = A(x) y + b(x)$, then all other solutions can be obtained by adding a solution to the homogeneous equation.

In this case $\tilde{y}$ is called a particular solution

The variation of constants formula

Let $Y(x)$ be a fundamental solution, and set $z(x) = Y(x)\, v(x)$, where $v(x)$ is an unknown vector. Then

$$z'(x) = Y'(x)\, v(x) + Y(x)\, v'(x)$$

$$= A(x)\, Y(x)\, v(x) + Y(x)\, v'(x).$$

We hope to find $z'(x) = A(x) z(x) + b(x)$

$$= A(x)\, Y(x)\, v(x) + b(x).$$

But this requires $Y(x)\, v'(x) = b(x)$, or (because $Y(x)$ is invertible)

$$v'(x) = Y^{-1}(x)\, b(x).$$

and hence $v(x) = v(x_0) + \int_{x_0}^{x} Y^{-1}(\xi)\, b(\xi)\, d\xi$

$$\Rightarrow \quad z(x) = Y(x)\, v(x_0) + \int_{x_0}^{x} Y(x)\, Y^{-1}(\xi)\, b(\xi)\, d\xi.$$

One can take e.g. the fundamental matrix X that satisfies $X(x_0) = I$.

Then we find

$$y(x) = X(x)\, y_0 + \int_{x_0}^{x} X(x)\, X^{-1}(\xi)\, b(\xi)\, d\xi$$

to be the solution to $\begin{cases} y' = A(x)y + b(x) \\ y(x_0) = y_0 . \end{cases}$

Note because any fundamental solution can be written $Y(x) = X(x)\, Y(x_0)$

$$\Leftrightarrow \quad X(x) = Y(x)\, Y^{-1}(x_0)$$

it also is true that

$$y(t) = Y(x)\, Y^{-1}(x_0)\, y(x_0) + \int_{x_0}^{x} Y(x)\, Y^{-1}(\xi)\, b(\xi)\, d\xi .$$

Positive solutions

Many models using ordinary differential equations require positive solutions. This can be population models, models for chemical reactions etc. Populations and chemical concentrations are non-negative.

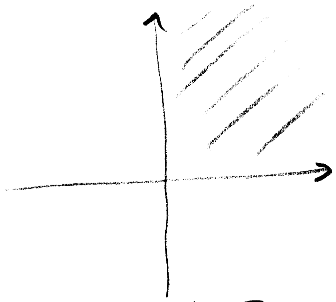
Def A matrix $A(t) = (a_{ij}(t))$ is said to be essentially positive if $a_{ij}(t) \geq 0$ when $i \neq j$.

Theorem If $A(t)$ is essentially positive and

$$\begin{cases} \frac{du}{dt} \geq A(t)u \\ u(t_0) \geq 0 \end{cases}$$

(meaning that all coordinates are non negative),

then $u(t) \geq 0$ for all t. We say that $u(t)$ belongs to the positive cone



the positive cone of $\mathbb{R}^2$.

Proof Let $B(t) = A(t) + h(t)I$, where $h(t) = \max_{1 \leq i \leq n} (-a_{ii}(t))$. Then $B(t) = (b_{ij}(t))$, where all elements $b_{ij}(t)$ are non-negative.

Then $\frac{du}{dt} + h(t)u(t) \geq B(t)u(t)$

That means that each component $u^i(t)$ of $u(t)$ satisfies

$\star$ $\frac{du^i}{dt} + h(t)u(t) \geq 0$ at $t = t_0$ and hence that

$\frac{d}{dt}\left(e^{\int_{t_0}^{t} h(t)dt} u(t)\right) \geq 0$ at $t = t_0$, thus it is non decreasing,

and $\star$ remains true for all t, and so $u(t) \geq 0$ for all t $(\geq t_0)$

Systems with constant coefficients

Functions of exponentials.

Recall that $x' = ax$ has the solution $x(t) = e^{at} x_0$ (in $\mathbb{R}$)

What about $x' = Ax$ in $\mathbb{R}^n$? Can we write
$x(t) = e^{At} x_0$ in $\mathbb{R}^n$, where A is an $n\times n$-matrix?

Yes. Consider the matrix equation

$$\begin{cases} X'(t) = A X(t) \\ X(t_0) = I \end{cases}$$

The Picard method gives

$$X_0 = I$$

$$X_1(t) = I + \int_{t_0}^{t} A X_0(\tau)\, d\tau = I + \int_{t_0}^{t} A I \, d\tau = I + At$$

$$X_2(t) = I + \int_{t_0}^{t} A(I + A\tau)\, d\tau = I + A\tau + A^2 \frac{t^2}{2}$$

$$\vdots$$

$$X_n(t) = I + At + \frac{A^2 t^2}{2} + \dots + \frac{A^n t^n}{n!}$$

$$\Rightarrow \quad X(t) = I + At + \dots + \frac{A^n t^n}{n!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

So at least formally $X(t)$ is a powerseries in At with the same coefficients as e^t, and so it is natural to write

$$X(t) = e^{At}.$$

Note also that this series is convergent, because

$$\| X_n(t) \| \leq 1 + \sum_{k=1}^{n} \frac{\| (At)^k \|}{k!} \leq$$

$$\leq 1 + \sum_{k=1}^{n} \frac{\| A \|^k t^k}{k!} \leq e^{\| At \|}$$ (the usual exponential).

(recall that $\| AB \| \leq \| A \| \| B \|$, and so
$\| A^k \| \leq \| A \|^k$).

General powerseries of matrices

Let $p(s)$ be a (complex) polynomial of one variable s, so $p(s) = c_0 + c_1 s + \ldots + c_k s^k$.

Then we define
$$p(A) = c_0 I + c_1 A + c_2 A^2 + \ldots + c_k A^k,$$
and $p(tA) = c_0 I + c_1 A t + c_2 A^2 t^2 + \ldots + c_k A^k t^k$.

and similarly for (formal) powerseries.

Lemma a) if $BC = CB$ (the matrices A and B commute)

then $e^{B+C} = e^B e^C$

b) If $\det C \neq 0$, then
$$e^{C^{-1} B C} = C^{-1} e^B C.$$

c) Let $\operatorname{diag}(\mu_1 \ldots \mu_n) = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$. We have

$$e^{\operatorname{diag}(\lambda_1 \ldots \lambda_n)} = \operatorname{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}).$$

Proof

a) $$e^{B+C} = \sum_{k=0}^{\infty} \frac{1}{k!} (B+C)^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} B^j C^{k-j}$$

(this is where we use $BC=CB$. For example
$(B+C)^2 = (B+C)(B+C) = B^2 + BC + CB + C^2$
$= B^2 + 2BC + C^2$)

$$= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{1}{(k-j)!} \frac{1}{j!} B^j C^{k-j}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{j!} B^j C^k = \left(\sum_{j=0}^{\infty} \frac{1}{j!} B^j \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} C^k \right)$$

Not that this is allowed because each series is absolutely convergent.

b) Note that $(C^{-1}BC)^k = C^{-1}BC\,C^{-1}BC\,C^{-1}BC \ldots C^{-1}BC$
$$= C^{-1} \underbrace{B \; B \; B \ldots B}_{k \text{ times}} C = C^{-1} B^k C.$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{k!} (C^{-1}BC)^k = C^{-1} \left(\sum_{k=0}^{\infty} \frac{1}{k!} B^k \right) C = C^{-1} e^B C.$$

c) $$(\operatorname{diag}(\lambda_1 \ldots \lambda_n))^k = \operatorname{diag}(\lambda_1^k, \ldots \lambda_n^k).$$

Corollaries
$$(e^A)^{-1} = e^{-A}$$

$$e^{A(s+t)} = e^{As} e^{At}$$

$$e^{A+\lambda I} = e^A e^\lambda$$

We have seen that $X(t) = e^{At}$ is a fundamental matrix to the equation
$$y' = Ay.$$
But it seems hard to evaluate the powerseries since it involves many matrix multiplications.