Linear systems with constant coefficients

Consider

$$(\circledast) \quad y' = Ay.$$

Since the scalar equation $y' = ay$ has solution $y = e^{at} y(t_0)$, try a solution of the same form for the system:

Ansatz $\quad y(t) = e^{\lambda t} c \quad$ for some λ and vector c.

Then $\quad y'(t) = \lambda e^{\lambda t} c \quad$ and then

$$\lambda e^{\lambda t} c = A e^{\lambda t} c \iff Ac = \lambda c.$$

That means that λ is an eigenvalue to A, and c an eigenvector to λ.

<u>Def</u> $\quad p_n(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial to A, and the n zeros (counted with multiplicity) are the eigenvalues.

<u>Theorem</u> $\quad y = e^{\lambda t} c$ is a (non trivial) solution to $\circledast$ if and only if λ is an eigenvalue and c an eigenvector.

If the eigenvalues $\lambda_1 \dots \lambda_n$ are different, then

$e^{\lambda_1} c_1, \dots, e^{\lambda_n} c_n$ are independent solutions, and

so $\quad Y(t) = \left[e^{\lambda_1} c_1 \;\; e^{\lambda_2} c_2 \;\; \dots \;\; e^{\lambda_n} c_n \right]$ is a fundamental matrix.

<u>Proof</u> We only need to recall that eigenvectors corresponding to different eigenvalues are independent:

If $\quad Ac_j = \lambda_j c_j \quad j = 1 \dots k, \quad \lambda_j \neq \lambda_i$ if $i \neq j$, then if

$$\sum_{j=1}^{k} \alpha_j c_j = 0 \Rightarrow \sum_{j=1}^{k} \alpha_j A c_j = 0 \Rightarrow \sum_{j=1}^{k} \alpha_j \lambda_j c_j = 0$$

we want to show that $\sum \alpha_j c_j = 0$ and $\sum \alpha_j \lambda_j c_j = 0 \Rightarrow \alpha_1 = \dots = \alpha_k = 0$.

This is obviously true for $k=1$. Suppose it holds for $k=k_0$. Then with $k=k_0+1$, assume that
$$\sum_{j=1}^{k_0+1} \alpha_j c_j = 0 \quad \text{and} \quad \sum_{j=1}^{k_0+1} \alpha_j \lambda_j c_j = 0 .$$
and that all $\alpha_j \neq 0$. Then also $\sum_{j=1}^{k_0+1} \alpha_j \lambda_{k_0+1} c_j = 0$

and therefore $\sum_{j=1}^{k_0+1} \alpha_j (\lambda_j - \lambda_{k_0+1}) c_j = 0$
$$\Rightarrow \sum_{j=1}^{k_0} \alpha_j (\lambda_j - \lambda_{k_0+1}) c_j = 0 .$$
But $\lambda_j - \lambda_{k_0+1} \neq 0$,

and therefore $\sum_{j=1}^{k_0} \beta_j c_j = 0$, and multiplying by A,
$$\sum_{j=1}^{k_0} \beta_j \lambda_j c_j = 0 .$$
By the induction hypothesis this implies $\beta_j = 0$ $\quad j=1,\dots,k_0$ and because $\alpha_j = \dfrac{\beta_j}{\lambda_j - \lambda_{k_0+1}}$ also $\alpha_j = 0$ $(j=0\dots k_0)$. Then obviously also $\alpha_{k_0+1} = 0$ and the result follows by the induction principle.

Note that here the λ_i and also the eigenvectors may be complex, but we really look for real solutions to $y' = Ay$.

Theorem (real version) Let $\lambda = \mu + i\nu$ be a complex eigenvalue, and $c = a + ib$ a complex eigenvector. Then $y = e^{\lambda t} c$ provides two real solutions $u = \mathrm{Re}\, y$ and $v = \mathrm{Im}\, y$. These are independent solutions.

Proof If $Ac = \lambda c$, then $A\bar{c} = \bar{\lambda} \bar{c}$, and so $\bar{\lambda}$ is an eigenvector with eigenvector. c and $\bar{c}$ are independent because $\lambda \neq \bar{\lambda}$ and so also u and v must be independent.

Another example that can be treated rather easily is when A is a symmetric matrix. Then there is an orthogonal matrix C such that $C^T A C = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, where the λ_i are eigenvalues of A.

The λ_i's are in this case real.

The fundamental matrix

$X = e^{At}$ can then be computed easily:

$$C^T X C = e^{C^T A C t} = e^{\mathrm{diag}(\lambda_1 \dots \lambda_n)} = \mathrm{diag}(e^{\lambda_1}, \dots e^{\lambda_n}),$$

because $C^{-1} = C^T$. So

$$X(t) = C \,\mathrm{diag}(e^{\lambda_1} \dots e^{\lambda_n})\, C^T$$

Note that in the two caseses, if in

$$y' = Ay,$$

we write $y = Cz$, we find

$$\frac{d}{dt}(Cz) = ACz \quad \text{and then}$$

$$\frac{d}{dt}(C^{-1}Cz) = C^{-1}ACz$$

$$z' = C^{-1}ACz = \cancel{\quad}\ z$$

$$= C^{-1}\begin{pmatrix} | & | & & | \\ \lambda_1 c_1 & \lambda_2 c_2 & \dots & \lambda_n c_n \\ | & | & & | \end{pmatrix} z$$

$$= \mathrm{diag}(\lambda_1 \dots \lambda_n)\, z$$

This essentially solves our problem in case of n <u>different</u> eigenvalues, or for symmetric matrices.

The Jordan form

Let A be a (complex) $n \times n$ - matrix.
There exists a nonsingular matrix C so that

$$B = C^{-1} A C$$

has the form

$$\begin{pmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{J_k} \end{pmatrix}$$

where $J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$, a matrix of type $n_i \times n_i$.

Each block J_i corresponds to one eigenvalue λ_i, and $\sum n_i = n$. The form of B corresponds to the characteristic polynomial of A,

$$p(\lambda) = \det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}.$$

Note that a differential equation

$y' = Ay$ is transformed to a simpler form

by writing $Cz = y \Rightarrow z' = C^{-1} A C z = Bz$.

If we let $z = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$, where z_i is a vector in $\mathbb{R}^{n_i}$,

we obtain k different systems:

$$\dot{z}_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix} z_i \quad , \text{ which we will analyse next.}$$

Note 2

The Jordan form corresponds to "invariant subspaces" of a matrix A.

An invariant subspace is by definition a linear subspace $\Lambda \subset \mathbb{R}^n$, so that for any $v \in \Lambda$, $Av \in \Lambda$. For a diagonalizable matrix, every eigenvector is the basis of a one-dimensional invariant subspace.

Prop. If λ is an eigenvalue to A, then $\{ v \in \mathbb{R}^n : Av = \lambda v \}$ is an invariant subspace.

Def A generalized eigenvector (to the eigenvalue λ) is a vector $\neq 0$ that satisfies

$$(A - \lambda I)^k v = 0 \qquad (k \geq 1)$$

Note that for a Jordan block,

$$(J - \lambda I)^k = \left(\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} - \lambda I \right)^k = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}^k$$

$= 0$ if $k \geq n_i - 1$.

Note 1 If the characteristic polynomial is

$$p(\lambda) = (-1)^n (\lambda-\lambda_1)^{n_1} \cdots (\lambda-\lambda_k)^{n_k},$$

we call n_i the algebraic multiplicity of the eigenvalue λ_i. The number of eigenvectors corresponding to λ_i must be a number r_i, $1 \le r_i \le n_i$. The number r_i is called the geometric multiplicity.

If $r_i = n_i$, then the Jordan blocks corresponding to λ_i is

$$\begin{pmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix},$$ so in fact it consists of n_i trivial blocks. The other extreme,

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix},$$ an $n_i \times n_i$-matrix

corresponds to the case with only one eigenvector.

The fundamental matrix to $\dot{z}_i = J_i z_i$ is

$$X_i = \begin{pmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{1}{2}t^2 e^{\lambda_i t} & \cdots \\ & \ddots & \ddots & \\ & & \ddots & te^{\lambda_i t} \\ & & & e^{\lambda_i t} \end{pmatrix}$$

(so X_i solves

$$\begin{cases} \dot{X}_i(t) = J_i X_i(t) \\ X_i(0) = I \end{cases}$$

The Cayley-Hamilton theorem

Theorem

Let A be an $n \times n$ - matrix, and let

$p_A(\lambda) = \det(A - \lambda I)$ be the characteristic

polynomial. Then $p_A(A) = 0$

Proof see special notes.

Ex $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \Rightarrow p_A(\lambda) = (1-\lambda)(2-\lambda) = 2 - 3\lambda + \lambda^2$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 9 & 4 \end{pmatrix}$$

$$2I - 3A + A^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 9 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A consequence is that any power of a matrix A $(n \times n)$ is a linear combination of $I, A, A^2, \dots, A^{n-1}$.

If $p_A(\lambda) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0)$, then

$$p_A(A) = 0 \Rightarrow A^n = -(a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \dots + a_1 A + a_0 I)$$

$$A^{n+1} = -(a_{n-1}A^n + a_{n-2}A^{n-1} + \dots + a_1 A^2 + a_0 A)$$

$$= (a_{n-1}^2 + a_{n-2})A^{n-1} + (a_{n-1}a_{n-2} - a_{n-3})A^{n-2} + \dots$$

$$+ (a_{n-1}a_1 - a_0)A + a_{n-1}a_0 I$$

etc.

This means that to compute powerseries it is enough compute small powers. But it is not trivial to compute the coefficients.

There is a better way.

Lemma

Let f be an analytic function, an p a polynomial of degree n. There is an analytic function g and a polynomial q of degree at most $n-1$ so that

$$f(z) = g(z)\,p(z) + q(z)$$

Remark so $q(z)$ is the "reminder" when $f(z)$ is divided by $p(z)$.

Proof by induction. Assume that $p(z) = z-c$

If $c=0$, then

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = z\left(\sum_{k=1}^{\infty} a_k z^{k-1}\right) + a_0$$

so the result holds with $q(z) = a_0$ and $g(z) = \sum_{k=0}^{\infty} a_{k+1} z^k$.

For a general c, let $w = z-c$, and let $F(w) = f(w+c)$. Then $F(w) = G(w)\,w + \tilde{a}_0$

$$\Leftrightarrow \quad f(z) = G(z-c)\,(z-c) + \tilde{a}_0 .$$

so the result is ok for $n=1$.

Suppose the lemma holds for $n = k-1$.

Let $p(z)$ be a polynomial of degree k.

Then $p(z) = p_1(z)\,(z-c)$, where $p(c)=0$.

By the induction hypothesis

$$f(z) = g_1(z)\,p_1(z) + q_1(z), \quad \text{where } \operatorname{degree}(q_1) \le k-2.$$

Also, $g_1(z)$ is analytic, and it can be written

$$g_1(z) = g(z)\,(z-c) + q_0 \qquad (q_0 \text{ a constant}).$$

Hence

$$f(z) = g_1(z)\, p_1(z) + q_1(z)$$
$$= \left(g(z)\,(z-c) + q_0\right) p_1(z) + q_1(z) =$$
$$= g(z)\,\underbrace{(z-c)\, p_1(z)}_{p(z)} + \underbrace{q_0\, p_1(z) + q_1(z)}_{q(z)}$$

we have proven that the result also holds for $n=k$, and by induction for all $n \geq 1$.

How do we use that?

If we wish to compute $f(A)$, where $f(z)$ is an analytical function, then we write

$$f(z) = g(z)\, p_A(z) + q(z) \quad \text{, and find that}$$

$$f(A) = g(A)\, p_A(A) + q(A) = q(A).$$

Ex $\quad A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \Rightarrow p_A(\lambda) = (\lambda-1)(\lambda-2)$

$$\sin(z) = g(z)\,(z-1)(z-2) + q(z) \qquad q(z) = q_0 + q_1 z$$

$$\Rightarrow \begin{matrix} \sin(1) = q(1) \\ \sin(2) = q(2) \end{matrix} \Rightarrow \begin{cases} q_0 + q_1 = \sin(1) \\ q_0 + 2q_1 = \sin(2) \end{cases}$$

$$\Rightarrow \begin{cases} q_1 = \sin(2) - \sin(1) \\ q_0 = \sin(1) - q_1 = 2\sin(1) - \sin(2) \end{cases}$$

$$\Rightarrow q(z) = 2\sin(1) - \sin(2) + (\sin(2) - \sin(1))\, z$$

$$\Rightarrow \sin(A) = (2\sin(1) - \sin(2))\, I + (\sin(2) - \sin(1))\, A$$

$$= \begin{pmatrix} 2\sin 1 - \sin 2 & 0 \\ 0 & 2\sin 1 - \sin 2 \end{pmatrix} + \begin{pmatrix} \sin 2 - \sin 1 & 0 \\ 3\sin 2 - 3\sin 1 & 2\sin 2 - 2\sin 1 \end{pmatrix} = \begin{pmatrix} \sin 1 & 0 \\ 3\sin 2 - 3\sin 1 & \sin 2 \end{pmatrix}$$

Linear systems of order n

(note that here the coefficients may depend on t)

A scalar equation of order n can be written

$$L(t, \tfrac{d}{dt})\, y = g, \quad \text{where}$$

$L(t,\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ is a (real) polynomial of degree n, and where

$L(t, \frac{d}{dt})$ means $\frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} \dots + a_1 \frac{d}{dt} + a_0$

Ex $\quad y'' + ay' + b = L(t, \frac{d}{dt})\, y,$

where $L(t, \frac{d}{dt}) = \lambda^2 + a\lambda + b.$

The equation

$$L(t, D)\, y = g$$

can be written $\quad x' = Ax + b$

where

$$x = \begin{pmatrix} y \\ y' \\ y'' \\ y^{(n-1)} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix}$$

and $b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}$. Note that all coefficients may depend on t.

Let $F(t)$ be a fundamental matrix to $x' = Ax + b$,

$$F(t) = \begin{pmatrix} y_1(t) & y_2(t) & & y_n(t) \\ y_1'(t) & y_2'(t) & & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{pmatrix} = \begin{pmatrix} R_1(t) \\ R_2(t) \\ \vdots \\ R_n(t) \end{pmatrix}$$

Note that the rows satisfy

$$R_{k+1} = R_k' \qquad (k=1,\dots,n-1)$$

We know that

$$x(t) = F(t)F^{-1}(0)x(0) + F(t)\int_0^t F^{-1}(s)\,b(s)\,ds$$

$$= F(t)F^{-1}(0)x(0) + $$

$$+ F(t)\int_0^t K_n(s)\,g(s)\,ds$$

↑ last column of $F^{-1}(s)$

If we are interested in $y(t)$ only, then we look for the first element of x, i.e.

$$y(t) = R_1(t)\,F^{-1}(0)\,x(0) + \int_0^t R_1(t)\,K_n(s)\,g(s)\,ds .$$

Note that the initial value problem requires information on $y(0), y'(0), \dots, y^{(n-1)}(0)$, and therefor the full vector $x(0)$.

<u>Def</u> $E(t,s) = R_1(t)K_n(s)$ is called the fundamental solution to $L(t,\frac{d}{dt})y = g$.

<u>Theorem</u> ⓐ $E(t,s)$ is the unique solution to

$$\begin{cases} L(t,\frac{d}{dt})u = 0 \\ u(s) = u'(s) = \dots = u^{(n-2)}(s) = 0 \\ u^{(n-1)}(s) = 1 \end{cases}$$

ⓑ
$$\begin{cases} L(t,\frac{d}{dt})y = g \\ y(t_0) = \dots = y^{(n-1)}(t_0) = 0 \end{cases}$$

has the unique solution

$$y(t) = \int_{t_0}^t E(t,s)\,g(s)\,ds .$$

Higher order equations with constant coefficients

$$Lu = \sum_{i=0}^{n} a_i u^{(i)}(t) = 0 \qquad a_n = 1, \quad a_i \text{ const.}$$

In the form of a system: $y_1 = u$
$\vdots$
$y_n = u^{(n-1)}$

$$\frac{dy}{dt} = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix} y$$

The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} -\lambda & 1 & & 0 \\ 0 & \ddots & -\lambda & 1 \\ -a_0 & & -a_{n-2} & -\lambda - a_{n-1} \end{pmatrix}$$

$$= -\lambda \det \begin{pmatrix} -\lambda & 1 & & \\ 0 & \ddots & 1 & \\ \vdots & & -\lambda & 1 \\ -a_1 & & & -a_{n-1}-\lambda \end{pmatrix} \longleftarrow (*)$$

$$+ (-1)^{n-1} (-a_0) \det \begin{pmatrix} 1 & & 0 \\ -\lambda & \ddots & \\ 0 & -\lambda & 1 \end{pmatrix} =$$

$$= (-1)^n a_0 - \lambda \det \begin{pmatrix} (*) \end{pmatrix} = \ \ldots \ =$$

$$= (-1)^n (a_0 + a_1\lambda + \ldots + a_n \lambda^n)$$

Theorem If λ_0 is a solution to $P(\lambda)=0$, multiplicity k

then $e^{\lambda_0 t}$, $te^{\lambda_0 t}$, $\dots$ $t^{k-1}e^{\lambda_0 t}$ are

independent solutions to $Lu=0$.

Proof $0 = L(e^{\lambda t}) = \sum a_i (\lambda^i e^{\lambda t}) = (-1)^n e^{\lambda t} p(\lambda)$

$\Rightarrow$ λ is a solution to $p(\lambda)=0$.

Note that $t^q e^{\lambda t} = \frac{d^q}{d\lambda^q} e^{\lambda t}$

$$\Rightarrow \quad L(t^q e^{\lambda}) = L\left(\frac{d^q}{d\lambda^q} e^{\lambda t}\right) = \frac{d^q}{d\lambda^q} L(e^{\lambda t})$$

$$= (-1)^n \frac{d^q}{d\lambda^q}\left(e^{\lambda t} p(\lambda)\right).$$

Here $P(\lambda) = Q(\lambda)(\lambda-\lambda_0)^k \Rightarrow L(t^q e^{t\lambda}) = 0$ if $\lambda=\lambda_0$, $q<k$.

We have to prove that they are independent, i.e.

if λ_i are solutions to $p(\lambda)=0$ mult k_i, and

$$\sum_{i=1}^{m} r_i(t) e^{\lambda_i t} = 0,$$

then $r_i(t)=0$.

Proof: clearly true if $m=1$.

Suppose it is true for $m=k-1$. Then

$$0 = \sum_{i=1}^{k} r_i(t) e^{\lambda_i t} = \sum_{i=1}^{k-1} r_i(t) e^{\lambda_i t} + r_k(t) e^{\lambda_k t}$$

$$\Rightarrow \quad \sum_{i=1}^{k-1} r_i(t) e^{(\lambda_i-\lambda_k)t} + r_k(t) = 0$$

If we differentiate $s = (\text{degree } r_k(t) + 1)$ times

we find that

$$\sum_{i=1}^{k-1} \left(\frac{d}{dt}\right)^s \left(r_i(t) e^{(\lambda_i-\lambda_k)t}\right) = 0$$

$$= e^{-\lambda_k t} \sum_{i=1}^{k-1} q_i(t) e^{\lambda_i t} \Rightarrow q_i(t)=0 \Rightarrow r_i(t)=0 \Rightarrow r_k(t)=0$$

$i=1,\dots,k-1$

Stability

Consider again

$$y' = Ay \qquad A \text{ a constant } n\times n\text{-matrix.}$$

Chose C so that $C^{-1}AC = J$, a Jordan matrix.

$$J = \begin{pmatrix} \ddots & & \\ & B_k & \\ & & \ddots \end{pmatrix} \qquad B_k = \underbrace{\begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}}_{n_k} \Big\}\, n_k$$

With $z = C^{-1}y \quad (y = Cz) \Rightarrow$

$$z' = Jz \;, \quad \text{and if} \quad z = \begin{pmatrix} \vdots \\ z_k \\ \vdots \end{pmatrix} \Big\}\, n_k$$

$$\dot{z}_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix} z_k$$

Recall that
$$e^{tB_k} = \begin{pmatrix} e^{\lambda_k t} & te^{\lambda_k t} & \cdots & \frac{1}{(n-1)!}t^{n-1}e^{\lambda_k t} \\ & \ddots & \ddots & \vdots \\ & & \ddots & te^{\lambda_k t} \\ 0 & & & e^{\lambda_k t} \end{pmatrix}$$

(this is what we may recall: the solutions are of the form $p(t)e^{\lambda_k t}$, where $p(t)$ has degree at most $k-1$)

Conclusion: if $\operatorname{Re}\lambda_k > 0$, then the solution is exponentially increasing

if $\operatorname{Re}\lambda_k < 0$ exponentially decreasing

if $\operatorname{Re}\lambda_k = 0$ polynomial

Consider a 2×2-system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic polynomial $p_A(\lambda) = \lambda^2 - S\lambda + D$

where $S = \operatorname{Tr} A = a_{11} + a_{22}$ $\quad D = \det A$

The zeros:

$$\lambda = \frac{1}{2}\left(S - \sqrt{S^2 - 4D}\right)$$

$$\mu = \frac{1}{2}\left(S + \sqrt{S^2 - 4D}\right)$$

Three cases:

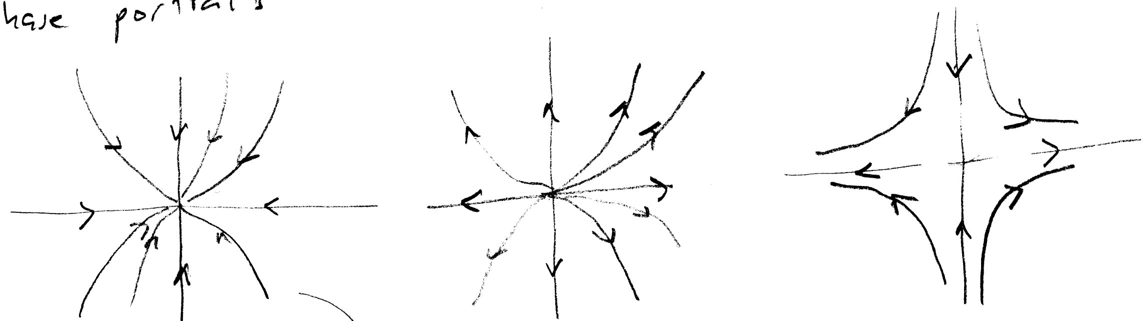
1) $A \longrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

2) $A \longrightarrow \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

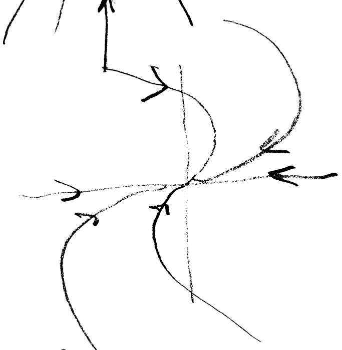
3) $A \longrightarrow \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix}$ $\quad (\lambda = \alpha + i\omega)$

Three phase portraits

A

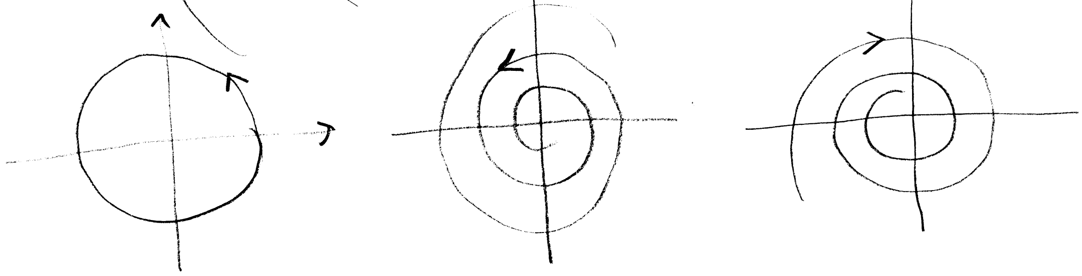


B



(only one eigen vector)

C



Definition Consider $y' = f(t,y)$ $\circledast$

A solution $x(t)$ to $\circledast$ is said to be stable (in the sense of Lyapunov) if

$\forall \varepsilon > 0, \exists \delta > 0$ so that

$|y(0) - x(0)| < \delta \Rightarrow |y(t) - x(t)| < \varepsilon$ for all $t \geq 0$.

("if $y(t)$ starts near $x(t)$ at $t=0$, then $y(t)$ and $x(t)$ remain close to each other for all t")

The solution $x(t)$ is said to be asymptotically stable if there is $\beta > 0$ so that if

$|y(0) - x(0)| < \beta$

then

$|y(t) - x(t)| \longrightarrow 0$

when $t \to \infty$.

Otherwise the solution is unstable otherwise.

Note Usually $x(t)$ is a stationary solution, i.e. $x(t) = \tilde{x}$ such that $f(t, \tilde{x}) = 0$.

Stability for linear systems

Theorem Consider $y' = A(t)y + b$ where $A(t)$ and $b(t)$ are continuous.

Suppose that $y(t) \equiv 0$ is a stable solution to the homogeneous equation, then every solution to ⊗ is stable.

Proof if

$$y' = A(t)y + b$$
$$x' = A(t)x + b,$$

then $(y-x)' = A(t)(y-x)$, or if $z = y-x$

$$z' = A(t)z.$$

Because $z=0$ is a stable solution to the homogeneous equation, then for all $\varepsilon > 0$ there is $\delta > 0$ so that

$$|z(0)| < \delta \Rightarrow |z(t)| < \varepsilon \quad \text{for all } t \geq 0.$$

Theorem If $\{\lambda_i\}$ are the eigenvalues of a constant matrix A, and if $\operatorname{Re}\lambda_i < \alpha$ for all i, then there is a constant C so that

$$|e^{At}| < Ce^{\alpha t}$$

($|\cdot|$ a compatible matrix norm)

Proof $y' = Ay$ has n independent solutions of the form $y = e^{\lambda t} p(t)$, where $p(t)$ is a polynomial of degree $\leq n$. If $\alpha - \operatorname{Re}\lambda_i = \varepsilon > 0$, then

$$|p_i(t)| < C_i e^{\varepsilon t} \Rightarrow$$

$$|e^{\lambda_i t} p_i(t)| \leq C_i e^{(\varepsilon + \operatorname{Re}\lambda_i)t} = C_i e^{\alpha t}.$$

A stability theorem for perturbations of linear systems with constant coefficients.

<u>Theorem</u> Consider the equation

$$y' = Ay + g(y,t) \qquad \text{where } g(y,t) \text{ is continuous}$$

for $t \geq 0$, $|y| < \alpha$ (some $\alpha > 0$).

Assume also that $\displaystyle\lim_{|y|\to 0} \frac{|g(y,t)|}{|y|} = 0$

uniformly in $t \geq 0$. Assume also that

$\operatorname{Re}\lambda < 0$ for all eigenvalues to λ.

Then $x(t) \equiv 0$ is an asymptotically stable solution to $y' = Ay + g(y,t)$.

<u>Proof</u> There are $C, \beta > 0$ so that

$$|e^{At}| \leq C e^{-\beta t} \qquad (\operatorname{Re}\lambda_i < -\beta).$$

Next, $\dfrac{|g(y,t)|}{|y|} \to 0$ when $|y| \to 0$ implies

that there is a $\delta > 0$ so that $0 < \delta < \alpha$ and

$$|g(z,t)| \leq \frac{\beta}{2C}|z| \quad \text{for all } t \geq 0, \ |z| \leq \delta.$$

We have

$$y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} g(y(s),s)\, ds$$

(this holds if $g(y(s),s)$ is replaced by a given function $b(s)$)

Then

$$|y(t)| \leq |e^{At} y_0| + \int_0^t |e^{A(t-s)} g(y(s),s)| \, ds$$

$$\leq C |y_0| e^{-\beta t} + \int_0^t C e^{-\beta(t-s)} \frac{\beta}{2C} y(s) \, ds$$

(if $|y(s)| < \delta$ for $0 \leq s \leq t$, we have to verify that afterwards!)

Let $|y_0| < \varepsilon$ and let $\phi(t) = |y(t)| e^{\beta t}$

Then
$$\phi(t) \leq C\varepsilon + \frac{\beta}{2} \int_0^t e^{-\beta t} \phi(s) \, ds$$
$$\leq C\varepsilon + \frac{\beta}{2} \int_0^t \phi(s) \, ds .$$

But then the Gronwall inequality implies
$$\phi(t) \leq C \varepsilon e^{\beta t/2} \Rightarrow |y(t)| \leq C \varepsilon e^{-\frac{\beta t}{2}}$$
if $|y(t)| < \delta$! That can be achieved letting $\varepsilon < \min(\alpha, \frac{\delta}{2C})$

<u>Theorem</u> Consider again $y' = Ay + g(t,y)$, where g satisfies the same conditions as in the previous theorem.

If at least one of the eigenvalues has a positive real part, then

$y \equiv 0$ is an unstable solution.

Proof First we note that we may assume that A is in the Jordan form, so that

$$A = \begin{pmatrix} \lambda_1 & k_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & k_{n-1} \\ 0 & & & \lambda_n \end{pmatrix}, \quad \text{where} \quad k_i = 0 \text{ or } k_i = 1.$$

A first problem to solve is that if $|\lambda_i|$ are very small, it may seem that the k_i may dominate.

But let $H = \begin{pmatrix} \eta & & 0 \\ & \eta^2 & \\ 0 & & \eta^n \end{pmatrix}$ (and so $H^{-1} = \begin{pmatrix} \eta^{-1} & & 0 \\ & \ddots & \\ 0 & & \eta^{-n} \end{pmatrix}$)

Then

$$H^{-1} A H = \begin{pmatrix} \lambda_1 & \gamma_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-1} \\ 0 & & & \lambda_n \end{pmatrix} \quad \text{where } \gamma_1 \dots \gamma_{n-1} = 0 \text{ or } \eta.$$

This is easily proven by writing $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} + \begin{pmatrix} 0 & k_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & k_{n-1} \\ 0 & & & 0 \end{pmatrix}$

so
$$H^{-1} A H = H^{-1} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} H + \begin{pmatrix} \eta^{-1} & & 0 \\ & \eta^{-2} & \\ 0 & & \eta^{-n} \end{pmatrix} \begin{pmatrix} 0 & k_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & k_{n-1} \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} \eta & & 0 \\ & \eta^2 & \\ 0 & & \eta^n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} + \begin{pmatrix} \eta^{-1} & & 0 \\ & \eta^{-2} & \\ 0 & & \eta^{-n} \end{pmatrix} \begin{pmatrix} 0 & k_1\eta^2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & k_{n-1}\eta^n \\ 0 & & & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} + \begin{pmatrix} 0 & k_1\eta & & 0 \\ & \ddots & \ddots & \\ & & \ddots & k_{n-1}\eta \\ 0 & & & 0 \end{pmatrix}.$$

Hence with $y = Hz$,

$$z' = \underbrace{H^{-1} A H}_{\tilde{A}} z + \underbrace{H^{-1} g(t, Hz)}_{f(t,z)}$$

We note that now $\tilde{A}$ has the same eigenvalues as A, and that
$$\frac{|f(t,z)|}{|z|} = \frac{|H^{-1} g(t, Hz)|}{|Hz|} \frac{|Hz|}{|z|}$$

But $|Hz| < |H|\,|z| \to 0$ when $|z| \to 0$ (but depending on η)

and $\dfrac{|H^{-1} g(t, Hz)|}{|Hz|} \le |H^{-1}| \dfrac{|g(t, Hz)|}{|Hz|} \to 0$ when $Hz \to 0$,

so $\dfrac{|f(t,z)|}{|z|} \to 0$ when $|z| \to 0$.

In fact, if $\eta < 1$, then $|H| = \eta$ and $|H^{-1}| = \eta^{-n}$.

so $$\frac{|f(t,z)|}{|z|} \le \frac{|H^{-1}|\ |g(t,Hz)|}{|Hz|} \frac{|Hz|}{|z|} \le \eta^{1-n} \frac{|g(t, Hz)|}{|Hz|}.$$

Given $\eta > 0$, choose δ_0 so that $\dfrac{|g(t,y)|}{|y|} < \eta^n$ when $|y| < \delta_0$. Then $\dfrac{|g(t, Hz)|}{|Hz|} < \eta^n$ when $|Hz| < \delta_0$,

but that is true if $|z| < \delta_0/\eta \equiv \delta$.

So for each $\eta > 0$, there is $\delta > 0$ so that

$$|z| < \delta \Rightarrow |f(t,z)| \le \eta |z|.$$

Componentwise we have

$$\dot{z}_j = \lambda_j z_j + \gamma_j z_{j+1} + f_j(t,z), \quad \text{where } \gamma_j = 0 \text{ or } \eta.$$

Let $\phi(t) = \sum_{\operatorname{Re}\lambda_j > 0} |z_j|^2$, $\psi(t) = \sum_{\operatorname{Re}\lambda_j \le 0} |z_j|^2$

so $|z|^2 = \phi(t) + \psi(t)$.

we shall prove that $\frac{d}{dt}(\phi - \psi) \geq \eta (\phi - \psi)$.

which implies instability, because then

$$\phi(t) - \psi(t) \geq e^{\eta t} (\phi(0) - \psi(0))$$

(we can choose initial data so that $\phi(0) - \psi(0) > 0$)

First,

$$\phi' = \sum_{I_1} \frac{d}{dt} |z_j|^2 = \sum_{I_1} 2 \operatorname{Re} z_j' \bar{z}_j \qquad I_1 = \{ j : \operatorname{Re} \lambda_j > 0 \}$$
$$I_2 = \{ j : \operatorname{Re} \lambda_j \leq 0 \}$$

$$= \sum_{I_1} 2 \operatorname{Re} \left(\lambda_j z_j \bar{z}_j + \gamma_j z_{j+1} \bar{z}_j + f_j \bar{z}_j \right)$$

$$= \sum_{I_1} 2 \operatorname{Re} \lambda_j |z_j|^2 + \sum_{I_1} 2 \gamma_j \operatorname{Re} (z_{j+1} \bar{z}_j) + \sum_{I_1} 2 \operatorname{Re}(f_j \bar{z}_j)$$

$$= A + B + C.$$

(note: we use the complex Jordan form, so solutions may be complex).

Let $\eta = \frac{1}{6} \min_{I_1} \operatorname{Re} \lambda_j \Rightarrow A \geq 2 \cdot 6 \eta \phi$

and $|B| \leq 2\eta \left| \sum_{I_1} \operatorname{Re}(z_{j+1}) \bar{z}_j \right| \leq 2 \eta \phi$

$$|C| \leq 2 \left| \sum_{I_1} \operatorname{Re}(f_j \bar{z}_j) \right| \leq 2 \sqrt{\phi} \sqrt{|f(t,z)|^2}$$
$$\leq \eta \phi + \frac{|f|^2}{\eta}.$$

Here we used the Cauchy inequality:

$$\left| \sum a_j b_j \right| \leq \sqrt{(\sum a_j^2)(\sum b_j^2)}$$

and the fact that

$$|ab| \leq \frac{1}{2}(a^2 + b^2) \quad \text{or} \quad |ab| = \left| a\sqrt{\eta} \frac{b}{\sqrt{\eta}} \right| \leq \frac{1}{2} \left(\eta a^2 + \frac{b^2}{\eta} \right).$$

We thus obtain

$$\phi' = A + B + C \geq 12\eta\phi - 2\eta\phi - \eta\phi - \frac{|f|^2}{\eta}$$
$$= 9\eta\phi - \frac{|f|^2}{\eta}.$$

If $|z| = \sqrt{\phi+\psi} \leq \delta$, then $|f|^2 \leq \eta^2(\phi+\psi)$

$$\Rightarrow \frac{|f|^2}{\eta} \leq \eta\phi + \eta\psi$$, and so, finally

$$\phi'(t) \geq 8\eta\phi - \eta\psi.$$

Next we need an estimate for ψ.

As before,

$$\psi' = \underbrace{\sum_{I_2} 2\,\mathrm{Re}\left(\lambda_j |z_j|^2\right)}_{\leq 0} + \underbrace{\sum_{I_2} 2\,\mathrm{Re}\,\sigma_j z_{j+1}\bar{z}_j}_{\leq 2\eta\psi} + \underbrace{\sum_{I_2} 2\,\mathrm{Re}\,(f\bar{z})}_{\leq \eta\psi + \eta(\phi+\psi)}$$

$$\leq 4\eta\psi + \eta\phi$$

But then

$$\frac{d}{dt}(\phi-\psi) \geq 8\eta\phi - \eta\psi - 4\eta\psi - \eta\phi$$
$$= 7\eta\phi - 5\eta\psi \geq 5\eta(\phi-\psi) \geq \eta(\phi-\psi)$$

$\blacksquare$

Note Why is $\left|\sum_{I_1} 2\,\mathrm{Re}\,\sigma_j z_{j+1}\bar{z}_j\right| \leq 2\eta\phi$, (and similarly $\left|\sum_{I_2} 2\,\mathrm{Re}\,\sigma_j z_{j+1}\bar{z}_j\right| \leq 2\eta\psi$)?

It is because of the structure of the Jordan matrices.

If $\sigma_j \neq 0$, then z_j and z_{j+1} correspond to the same Jordan block, and hence the same eigenvalue.