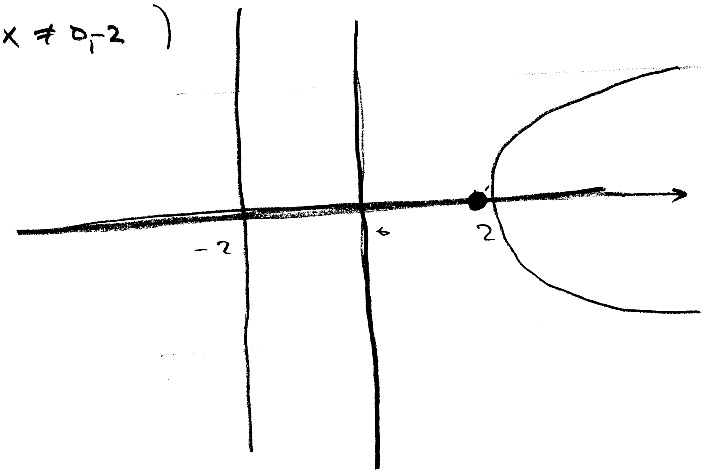
Ex Solve $y' = \dfrac{e^{-y^2}}{y\,(2x+x^2)}$ $\quad y(2)=0$

(well def if $y\neq 0$, $x\neq 0,-2$)



$$y e^{y^2} dy = \frac{1}{x(2+x)}dx$$

$$G(y)=\int_0^y \eta e^{\eta^2} d\eta = \frac{1}{2}\left(e^{y^2}-1\right)$$

$$F(x)=\int_2^x \frac{1}{\xi(\xi+2)}\,d\xi = \int_2^x \frac{1}{2}\left(\frac{1}{\xi}-\frac{1}{\xi+2}\right)d\xi$$

$$=\frac{1}{2}\Big[\ln|\xi|-\ln|\xi+2|\Big]_2^x = \frac{1}{2}\left(\ln\frac{|x|}{|x+2|}-\ln\frac{2}{4}\right)=\frac{1}{2}\ln\frac{2x}{x+2}$$

$$\Rightarrow \quad e^{y^2}-1=\ln\frac{2x}{x+2}$$

$$\Rightarrow \quad y^2=\ln\left(1+\ln\frac{2x}{x+2}\right)$$

$$y=\pm\sqrt{\ln\left(1+\ln\frac{2x}{x+2}\right)}$$

Remark

$$\begin{cases} y'(x)=f(x)\,g(y) \\ y(x_0)=y_0. \end{cases}$$

What happens if $g(y_0)=0$? Then $y(x)\equiv y_0$ is a solution

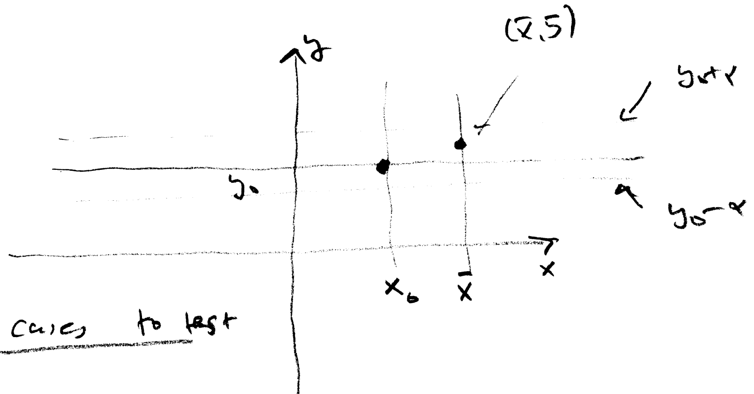
When is it unique?

Theorem If $\displaystyle\int_{y_0}^{y_0+\alpha}\frac{ds}{g(s)}$ and $\displaystyle\int_{y_0-\alpha}^{y_0}\frac{1}{g(s)}\,ds$ both are divergent,

then any solution starting above (below) $y=y_0$ remains above (below)

(and this implies uniqueness of $y(x)\equiv y_0$).

Proof



Several cases to test

Assume $y(x) \not\equiv y_0$,

and that $\bar{x} > x_0$ with $y_0 < y(\bar{x}) = \bar{y} < y_0 + \alpha$

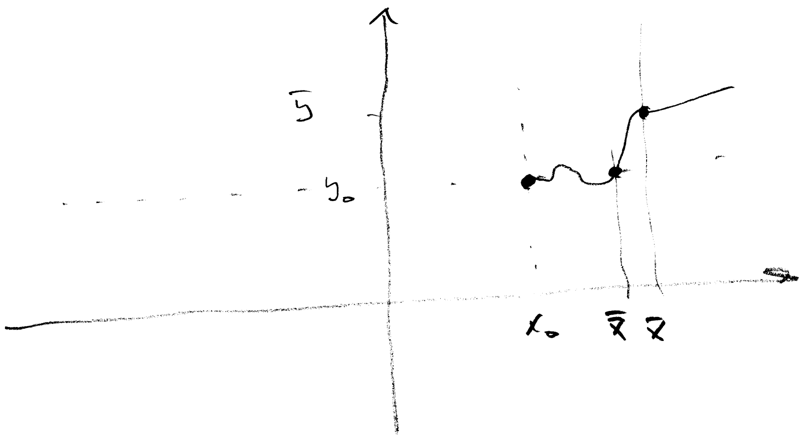
Then $\displaystyle\int_{\bar{y}}^{y(x)} \frac{ds}{g(s)} = \int_{\bar{x}}^{x} f(t)\,dt$

Let $y(\bar{\bar{x}}) < \bar{x}$ be the first point left of $\bar{x}$ such that $y(\bar{\bar{x}}) = y_0$ (this point exists if $y(x)$ is a solution with $y(x_0) = y_0$)

But then $\displaystyle\int_{\bar{y}}^{y_0} \frac{ds}{g(s)} = \int_{\bar{x}}^{\bar{\bar{x}}} f(t)\,dt$

divergent

convergent



Homogeneous equations

Note: "homogeneous" has a different meaning here ...

A function $f(x,y): \mathbb{R}^2 \to \mathbb{R}$ is said to be homogeneous of degree k if

$$f(tx, ty) = t^k f(x,y) \qquad (t \neq 0, \quad (x,y) \in \mathbb{R}^2 \setminus \{0\})$$

Ex for any $G(z): \mathbb{R} \to \mathbb{R}$,
$f(x,y) = G\left(\frac{x}{y}\right)$ is homogeneous of degree 0.

Ex Solve $y' = f\left(\frac{y}{x}\right)$

Ansatz: let $u = \frac{y(x)}{x}$. Then $y(x) = x u(x)$ and
$y'(x) = u(x) + x u'(x)$. Hence

$$u'(x) = \frac{1}{x}\left(y'(x) - u(x)\right) = \frac{1}{x}\left(f(u) - u\right), \quad \text{i.e.}$$

an equation with separated variables.

Ex Solve $y'(x) = \frac{y(x)}{x} - \frac{x^2}{y(x)^2}$, $y(1) = 1$

Then with $u(x) = \frac{y(x)}{x}$, $f(u) = u - \frac{1}{u^2}$

$$\Rightarrow \quad u'(x) = \frac{1}{x}\left(u - \frac{1}{u^2} - u\right) = -\frac{1}{x}\frac{1}{u^2}$$

$$\Rightarrow \quad u^2 u' = -\frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{u^3}{3}\right) = -\frac{1}{x}$$

$$\Rightarrow \quad \frac{u^3}{3} = -\ln|x| + C'$$

$$u(x) = \sqrt[3]{C - \ln|x|}.$$

$y(1) = 1 \Rightarrow u(1) = 1$
$\Rightarrow C = 1$

so $u(x) = \sqrt[3]{C - \ln|x|}$

Ex $$y'(x) = f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right) \quad (*)$$

(a particular example: $y'(x) = \frac{y+1}{x+2} - \exp\left(\frac{y+1}{x+2}\right)$,

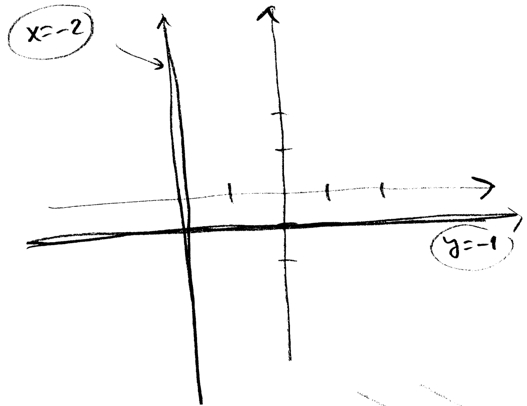
here $f(z) = z - e^z$

Solve $y+1=0$ $\quad x+2=0 \Rightarrow y_0=-1,\ x_0=-2$.

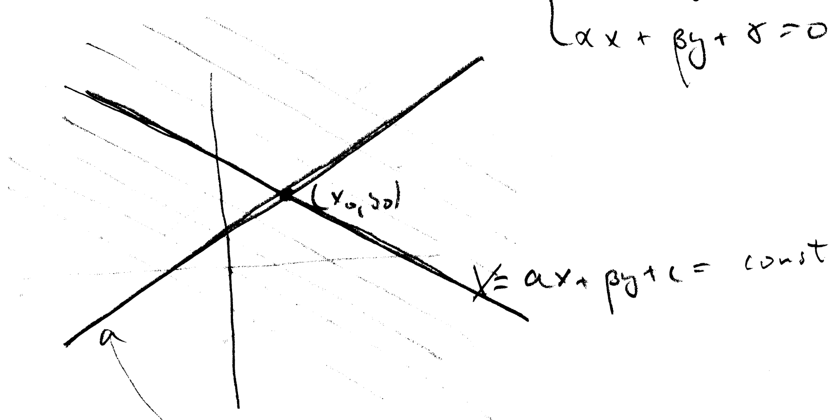
Then let $\bar{y} = y+1$, $\bar{x} = x+2$.

We get
$$\frac{d\bar{y}}{d\bar{x}} = \frac{dy}{dx} = f\left(\frac{y+1}{x+2}\right) = f\left(\frac{\bar{y}}{\bar{x}}\right) =$$

This is then a homogeneous equation



More generally:
solve
$$\begin{cases} ax+by+c=0 \\ \alpha x+\beta y+\gamma=0 \end{cases} \Rightarrow (x_0,y_0)$$



the change of variables $(x,y) \mapsto (X,Y)$ transforms equation $*$ to a homogeneous one.

Linear (first order, scalar) equations

$$y'(x) + g(x)\,y(x) = h(x)$$

g and h are given functions, continuous on an interval J.

Let $Ly(x) = y'(x) + g(x)\,y(x)$.

Then L is a linear operator:

$$L(y_1+y_2) = Ly_1 + Ly_2 \qquad y_1, y_2 \in C^1(J)$$

$$L(\alpha y) = \alpha L y \qquad \alpha \in \mathbb{R},\ y \in C^1(J)$$

Note $C^k(J) = \{ f : J \to \mathbb{R} : f, f', \dots f^{(k)}$ are continuous on $J \}$.

Homogeneous equations

Note here homogeneous means that $h(x) \equiv 0$, and consequently that $y(x) \equiv 0$ is a solution.

(B) $\begin{cases} y'(x) + g(x)\,y(x) = 0 \\ y(x_0) = y_0. \end{cases} \Rightarrow \dfrac{y'}{y} = -g$, a separable equation.

Hence $\dfrac{d}{dx} \log|y| = -g$

$$\Rightarrow \quad \log|y(x)| = -\int g(x)\,dx$$

With initial data,

$$\log|y| - \log|y_0| = -\int_{x_0}^{x} g(\xi)\,d\xi$$

$$\Rightarrow \quad y(x) = y_0\, e^{-\int_{x_0}^{x} g(\xi)\,d\xi}.$$

Alternatively: use integrating factor:

If $$Y(x) = e^{G(x)} y(x), \text{ then}$$

$$Y'(x) = e^{G(x)} y'(x) + G'(x) e^{G(x)} y(x)$$
$$= e^{G(x)} \left(y'(x) + G'(x) y(x) \right)$$

If $G(x) = \int g(x)\,dx$, then $G'(x) = g(x)$, and so the expression in ( ) is exactly the LHS of $\circledast$

This is the means of solving non-homogeneous equations:

$$y'(x) + g(x) y(x) = h(x) \quad \Leftrightarrow$$

$$e^{\int g(x)dx} \left(y'(x) + g(x) y(x) \right) = e^{\int g(x)dx} h(x) \quad \Leftrightarrow$$

$$\frac{d}{dx} \left(e^{\int g(x)dx} y(x) \right) = e^{\int g(x)dx} h(x).$$

or in more detail,

$$\frac{d}{dx} \left(e^{\int_a^x g(\zeta)d\zeta} y(x) \right) = e^{\int_a^x g(\zeta)d\zeta} h(x)$$

(the choice of a is arbitrary).

Then $$e^{\int_a^x g(\zeta)d\zeta} y(x) - e^{\int_a^{x_0} g(\zeta)d\zeta} y_0 = \int_{x_0}^{x} e^{\int_a^{\zeta} g(t)dt} h(\zeta)\,d\zeta$$

$$\Rightarrow \quad y(x) = \frac{e^{\int_a^{x_0} g(t)dt}}{e^{\int_a^x g(t)dt}} y_0 + \int_{x_0}^{x} \frac{e^{\int_a^{\zeta} g(t)dt}}{e^{\int_a^x g(t)dt}} h(\zeta)\,d\zeta$$

$$\Rightarrow \quad y(x) = e^{-\int_{x_0}^x g(t)dt} y_0 + \int_{x_0}^{x} e^{-\int_{\zeta}^{x} g(t)dt} h(\zeta)\,d\zeta.$$

Some non-linear equations that are in fact linear

A Bernoulli equation

$$(*)\quad \begin{cases} y'(x) + g(x)\,y(x) + h(x)\,y(x)^{\alpha} = 0 & (\alpha \neq 1) \\ y(x_0) = y_0 \end{cases}$$

g and h are continuous in an interval $J \subset \mathbb{R}$. $(y > 0)$

Then $(*) \iff$

$$(**)\quad \frac{y'}{y^{\alpha}} + g\,y^{1-\alpha} + h = 0$$

Note that

$$\frac{d}{dx} y^{1-\alpha} = (1-\alpha)\, y^{-\alpha}\, y'$$

Let $z(x) = y(x)^{1-\alpha}$

$$z'(x) = (1-\alpha)\, y^{-\alpha}\, y'$$

Then $(**)$ is the same as

$$\frac{1}{1-\alpha} z'(x) + g(x) z(x) + h(x) = 0$$

$$\iff \quad z'(x) + (1-\alpha)\, g(x)\, z(x) + (1-\alpha)\, h(x) = 0$$

This equation is linear in z, and can be solved by integrating factor.

<u>An exercise:</u> take $\alpha \geq 2$ $(\alpha \in \mathbb{N})$, $y \neq 0$. Show that for all x, $y(x) \neq 0$.

<u>Proof</u> take $z(x) = y(x)^{1-\alpha}$, $z' - (\alpha-1)\, g\, z = (\alpha-1)\, h$

$$\Rightarrow \quad z(x) = z(x_0)\, e^{(\alpha-1)\int_{x_0}^{x} g(t)\,dt} + (\alpha-1)\int_{x_0}^{x} e^{(\alpha-1)\int_{\xi}^{x} g(t)\,dt}\, h(\xi)\, d\xi.$$

then take x_0 so that $y(x_0) \neq 0$ (e.g. $y(x_0) > 0$)

we want to show that $y(x) \neq 0$ for all x

We have

$$z(x_0) = \frac{1}{y(x_0)^{\alpha-1}}$$

$\Rightarrow$ $|z(x)| < \infty$ for all $x \in \mathbb{R}$ (or J)

and then $y(x) = z^{\frac{1}{1-\alpha}}(x) \neq 0$. ☒

<u>Exercise</u> The Riccati equation
(important e.g. in the theory of optimal control)

$$y'(x) + g(x)\, y(x) + h(x)\, y(x)^2 = k(x).$$

Here g and k are in $C^0(J)$ $(J \subset \mathbb{R})$
and $h(x)$ in $C^1(J)$

There is a non-linear term, $h(x)\, y(x)^2$.

Let $u(x) = e^{\int h(x) y(x)\, dx}$

Then $u'(x) = h(x)\, y(x)\, e^{\int h(x) y(x)\, dx} = h(x)\, y(x)\, u(x)$

$$\begin{aligned} u'' &= h' y u + h y' u + h y u' \\ &= h' y u + h u(k - gy - hy^2) + h y h y u \\ &= h' y u + h u k - h u g y - h^2 u y^2 + h^2 u y^2 \\ &= h' y u + h u k - h u g y \\ &= \frac{h'}{h} u' + k h u - g u' \end{aligned}$$

So the Riccati equation ⊛ is equivalent to the second order linear equation

$$\boxed{u'' + \left(g - \frac{h'}{h}\right)u' - hk\,u = 0}$$

Non-constant coefficients, ..., cannot always be solved explicitly.

Suppose one solution is known. Then all can be found: Suppose that y and ϕ are two solutions. Then $u = y - \phi$ satisfies

$$u' = y' - \phi' = -g(y-\phi) - h(y^2 - \phi^2)$$

$$= -g(y-\phi) - h(y-\phi)(y+\phi)$$

$$= -gu - hy\,(y+2\phi) \quad , \text{ i.e.}$$

$$u' + (g + 2\phi h)\,u + hu^2 = 0.$$

This is a Bernoulli equation. ($\kappa = 2$) that can be solved.