Lyapunov functions and the Lyapunov stability theory

Def The solutions to $y' = f(y)$ are said to be exponentially stable if there exist β, γ and c (>0) so that

$$|y(0)| < \beta \Rightarrow |y(t)| < c e^{-\gamma t} \qquad (t > 0).$$

Note If f is locally Lipschitz, then exponential stability implies asymptotic stability. This can be proven as follows:
For $\varepsilon > 0$, there is $a \geq 0$ so that $c e^{-a\gamma} < \varepsilon$, so if $|y_0| < \beta$, then if $y(t, y_0)$ is the solution with $y(0) = y_0$, satisfies $|y(t, y_0)| \leq \varepsilon e^{-\gamma(t-a)}$ in $[a, \infty[$.
Also, because of the Lip cond on f, there is $\delta < \beta$ so that if $|y_0| < \delta$, $|y(t, y_0)| < \varepsilon$ when $t \in [0, a[$

Consider now $V \in C^1(D, \mathbb{R})$ where $D \subset \mathbb{R}^n$, $y_0 \in D$.
and let

$$\dot{V}(x) = \operatorname{grad} V(x) \cdot f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x) f_i(x),$$

i.e. the directional derivative of V in the direction of f.

Note that if $y(t)$ solves $y'(t) = f(y)$ then

$$\frac{d}{dt} V(y(t)) = \dot{V}(y(t)).$$

Definition A Lyapunov function to $y' = f(y)$ is a function $V : \mathbb{R}^n \to \mathbb{R}$ (or $V : D \subseteq \mathbb{R}^n \to \mathbb{R}$) such that

- $V \in C^1(D)$
- $V(x) > 0$ for $x \neq 0$
- $\dot{V}(x) \leq 0$ in $D \subseteq \mathbb{R}^n$.

Theorem Let $f \in C(D)$, $f(0) = 0$, and assume that there is a Lyapunov function to f. Then

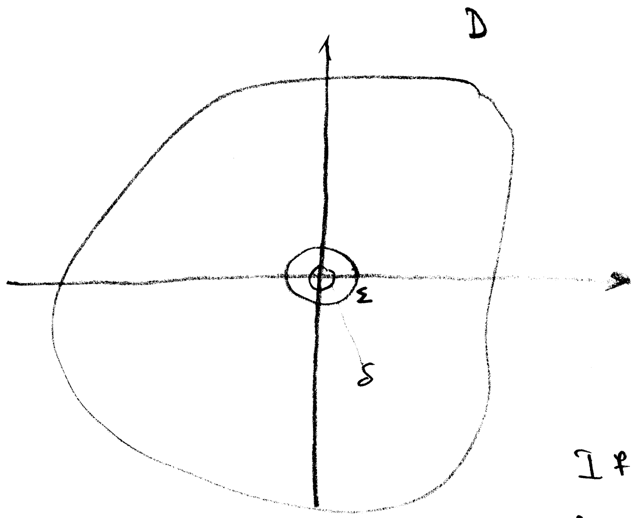
a) if $\dot{V} \leq 0$ in D, then $y \equiv 0$ is a stable solution to $y' = f(y)$

b) if $\dot{V} < 0$ in $D \setminus \{0\}$ then $y \equiv 0$ is asymptotically stable

c) if $\dot{V} \leq -\alpha V$ and $V(x) \geq b|x|^\beta$ in D (α, β, b positive), then $y \equiv 0$ is an exponentially stable solution.

Proof

a)



$\dot{y} = f(y)$, $f(0) = 0$

$$\frac{d}{dt} V(y(t)) = \dot{V}(y) \leq 0.$$

Take γ: $V(y) > \gamma$ on $|y| = \varepsilon$

Take $\delta \in]0, \varepsilon[$ so that $V(y) < \gamma$ in $|y| < \delta$.

If $|y(0)| < \delta$, we know

$$\dot{V}(y) = \frac{d}{dt} V(y(t)) \leq 0 \Rightarrow$$

$V(y(t)) < \gamma$ for all $t > 0$.

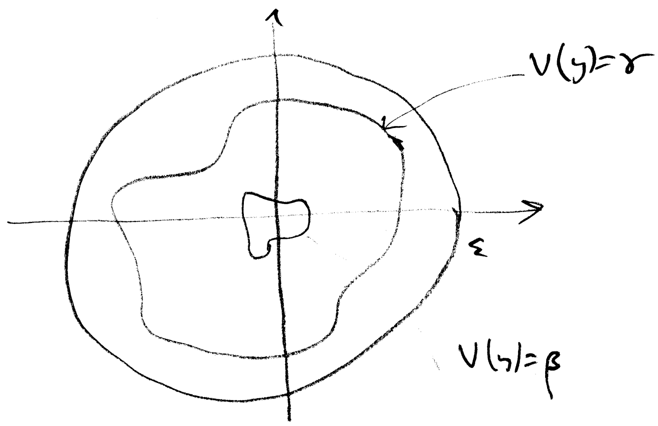
But then $|y(t)| < \varepsilon$ for all $t \geq 0$ because $y(t)$ is continuous.

b) Take $y(t)$ as in a), and let $\phi(t) = V(y(t))$.

Then $\frac{d}{dt}\phi < 0 \Rightarrow \phi(t) \to \beta < \gamma$ when $t \to \infty$

(because ϕ is strictly decreasing). Want to prove that $\beta = 0$.

If $\beta > 0$, let $M = \{ y : |y| \le \varepsilon,\ \beta \le V(y) \le \gamma \}$



Then M is a closed and bounded subset of $\{ 0 < |y| \le \varepsilon \}$.

Let $-\alpha = \max\left(\{ \dot{V}(y),\ y \in M \} \right)$.

We have $-\alpha < 0$.

If $y(t)$ remains in M,

then $\phi'(t) \le -\alpha \Rightarrow \phi(t) \le \phi(0) - \alpha t$

which contradicts the statement $\phi(t) \geq 0$.

This implies $y(t) \to 0$, because let

$\delta = \min\{ V(y) : \varepsilon_1 \le y \le \varepsilon \}$. $V(y(t)) \to 0 \Rightarrow \exists t_0 :$ $|y(t)| < \varepsilon_1$ for $t > t_0$.

c) We have in this case $b|y(t)|^\beta \le V(y(t)) = \phi(t)$

and $\phi' \le -\alpha \phi \Rightarrow \phi(t) \le \phi(0) e^{-\alpha t}$

$\Rightarrow |y(t)| \le \left(\frac{\phi(t)}{b}\right)^{1/\beta} \le \left(\frac{\phi(0)}{b}\right)^{1/\beta} e^{-\frac{\alpha}{\beta}t}$.

Theorem (instability)

Let $V \in C^1(D)$ satisfy $V(0)=0$ and assume that there is a sequence $\{y_k\}$, $0<|y_k| \to 0$ when $k\to\infty$ such that $V(y_k)>0$.

If $\dot V > 0$ for $|y| \neq 0$, or $\dot V \geq \lambda V$ $(\lambda>0)$,

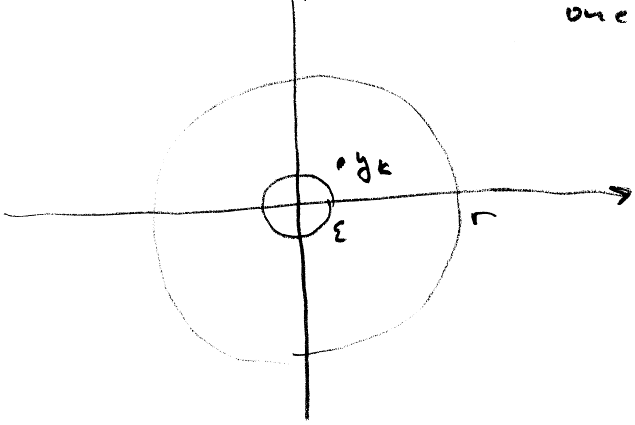
then $y(t)\equiv 0$ is an unstable solution.

Proof Let $y(t)$ solve $\begin{cases} y' = f(y) \\ y(0)=y_k \end{cases}$ Then

$\phi(0) = \alpha > 0$, where $\phi(t) = V(y(t))$. If $\dot V > 0$ for $|y|\neq 0$, take $\varepsilon>0$ so that $V(y)<\alpha$ for $|y|<\varepsilon$. We have $\phi' > 0$, and so $\alpha = \phi(0) \leq \phi(t) \Rightarrow y(t) > \varepsilon$.

Take $r > \varepsilon$ so that $\{|x|\leq r\} \subset D$. If $\varepsilon \leq |y| \leq r$, then $\dot V(y) \geq \beta > 0$ ($\dot V$ cont) $\Rightarrow \phi' \geq \beta \Rightarrow \phi(t) \geq \alpha + \beta t$.

But $V(y)$ is bounded when $|y|\leq r \Rightarrow y(t)$ must leave the ball $\{|y|\leq r\}$. Note that y_k can be chosen arbitrarily close to 0, and hence the result holds independently of how close to the origin one starts.



If $\dot V \geq \lambda V$ then $\phi' \geq \lambda \phi \Rightarrow y(t)$ must leave any ball $\{|y|\leq r\} \subset D$.

An example

$$u''(t) + h(u(t)) = 0$$

$$\Longleftrightarrow \quad \begin{cases} x(t) = u(t) \\ y(t) = u'(t) \end{cases}, \qquad \begin{cases} y' = -h(x) \\ x' = y \end{cases}$$

take $E(x,y) = \frac{1}{2} y^2 + \int_0^x h(s)\,ds$

$$= \frac{dE}{dt} = \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial y}\dot{y} = y(-h(x)) + h(x)y = 0$$

conservative

$$u''(t) + \varepsilon u'(t) + h(x) = 0$$

$$\Longleftrightarrow \quad \begin{aligned} x' &= y \\ y' &= -h(x) - \varepsilon y \end{aligned}$$

$$\Rightarrow \dot{E} = h(x)y + y(-h(x)) - \varepsilon y^2 = -\varepsilon y^2$$

dissipative

More examples

$$y' = Ay + \varphi(y) By + g(y)$$

$\varphi : \mathbb{R}^n \to \mathbb{R}$ $(D \to \mathbb{R})$ $\qquad g : \mathbb{R}^n \to \mathbb{R}^n, \quad g(0) = 0$

$$B = -B^T$$

Take $V(y) = y \cdot y = |y|^2$

$$\dot{V}(y) = 2y\dot{y} = 2y \cdot Ay + \varphi(y)\, 2y \cdot By + 2 \cdot y \cdot g(y)$$

$$= 2\langle y, Ay\rangle + 2\langle y, g(y)\rangle$$

need to find out if $\langle y, Ay\rangle < 0$ (A negative definite ..

$\langle y, g(y)\rangle < 0$

all eigenvalues negative)

2-point boundary value problems

$$u'' + a_1(x)\, u' + a_0(x)\, u = g(x) \qquad a \le x \le b$$

$(0 \le x \le 1)$

boundary cond

I $\quad u(a) = \eta_1 \quad u(b) = \eta_2 \quad$ (Dirichlet)

II $\quad u'(a) = \eta_1 \quad u'(b) = \eta_2 \quad$ (Neumann)

III $\quad \alpha_1 u(a) + \alpha_2 u'(a) = \eta_1 \quad$ (mixed)

Sturm-Liouville problems

Let
$$\begin{cases} Lu(x) = (p(x)\, u'(x))' + q(x) u(x) \\ R_1 u = \alpha_1 u(a) + \alpha_2\, p(a)\, u'(a) \\ R_2 u = \beta_2 \ \ldots\ldots \end{cases}$$

where $p \in C^1(J) \quad q, g \in C(J) \qquad J = [a,b]$

$p(x) > 0 \quad , \quad \alpha_1^2 + \alpha_1^2 > 0, \quad \beta_1^2 + \beta_2^2 > 0$

A boundary value problem

$$\begin{cases} Lu = g & \text{in } J \\ R_1 u = \eta_1 \quad R_2 u = \eta_2 \end{cases}$$

Note with $\begin{cases} y_1 = u \\ y_2 = p u' \end{cases}$ we can write

$$\begin{cases} y_2' + q y_1 = g \\ y_1' = y_2 / p \end{cases}$$ with bc

$\alpha_1 y_1(a) + \alpha_2 y_2(a) = \eta_1$

Note any system of the form

$u'' + a_1 u' + a_0 u$ can be written in the "self adjoint" form: Let $p(x) = e^{\int a_1 dx}$

Then $pu'' + a_1 pu' + pa_0 u =$

$= (pu')' + pa_0 u$

Lagranges identity

1) If $u, v \in C^2(J)$, then

$vLu - uLv = (p(u'v - v'u))'$

2) If $R_i u = R_i v = 0$ $(i = 1, 2)$ then

$\int_a^b (vLu - uLv)\, dx = 0$

Proof a direct calculation:

1) $v((pu')' + qu) - u((pv')' + qv) =$

$= v(pu')' - u(pv')' = p'(vu' - uv') + p(vu'' - uv'')$

$= p'(vu' - uv') + p(vu'' + v'u' - v'u' - uv'')$

$= (p(vu' - uv'))'$

2) $\int_a^b (vLu - uLv)\, dx = \int_a^b \frac{d}{dx}(p(vu' - uv'))\, dx$

$= \left[p(x)(v(x)u'(x) - u(x)v'(x))\right]_a^b = 0$ if R_i are as stated.

The boundary value problem is linear $\Rightarrow$

a) if u_j $j=1,\dots,n$ solve the hom. problem then $\sum c_j u_j$ also is a solution

b) if v_1, v_2 solve the (inhom) probl., then $v_1 - v_2$ solves the hom. problem.

c) one can add any solution of the hom problem to a solution of the inhom probl. and still have a solution

d) Let v^* solve the inhomogeneous problem. Then all solutions to the inhom probl. can be found by adding a solution to the hom. problem

<u>Theorem</u> Let u_1, u_2 be a fundamental system of solutions to $Lu=0$. Then the inhom boundary value probl. can be solved uniquely if and only if the homogeneous problem has only the solution $u \equiv 0$. This holds iff

$$\det \begin{pmatrix} R_1 u_1 & R_1 u_2 \\ R_2 u_1 & R_2 u_2 \end{pmatrix} \neq 0 .$$

Compare the linear (algebraic) system

$$A x = b \qquad \begin{array}{l} A \quad n \times n \\ x, b \in \mathbb{R}^n \end{array}$$

this has a unique soln $\Leftrightarrow$ $\det A \neq 0$

$\Leftrightarrow$ $Ax=0$ has only $x=0$ as solution.

"The Fredholm alternative"

Proof The difference w of two solutions to

$$\begin{cases} Lu = g \\ R_1 u = \eta_1 \quad R_2 u = \eta_2 \end{cases}$$

solves $\begin{cases} Lw = 0 \\ R_1 w = R_2 w = 0 \end{cases}$

which gives the uniqueness statement.

Take any solution v_* to $Lu = g$.

The general solution is

$$V = v_* + c_1 u_1 + c_2 u_2$$

Check b.c.:

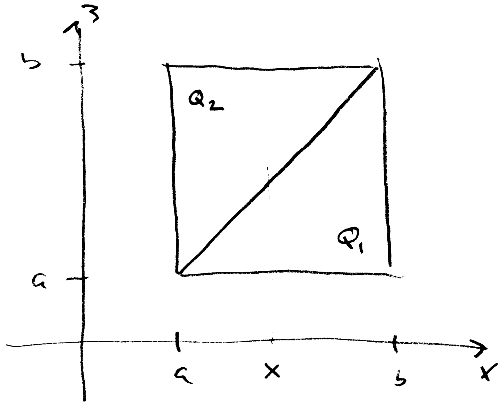
$$R_i v = R_i v_* + c_1 R_i u_1 + c_2 R_i u_2 = \eta_i$$

The eq.

$$\begin{cases} R_1 u_1 \, c_1 + R_1 u_2 \, c_2 = \eta_1 - R_1 v_* \\ R_2 u_1 \, c_1 + R_2 u_2 \, c_2 = \eta_2 - R_2 v_* \end{cases}$$

has a unique soln. $\Leftrightarrow$ $\det(\quad) \neq 0$.

Fundamental solutions



$Q = Q_1 \cup Q_2$

Q_1, Q_2 **closed** triangles

A function $\gamma : Q \to \mathbb{R}$ is called a fundamental solution to $Lu = 0$ if

a) $\gamma(x, \xi)$ is cont in Q

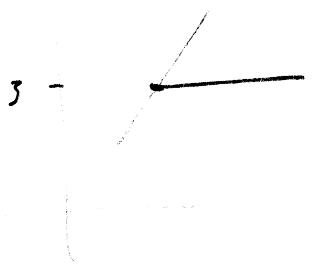
b) $\frac{\partial \gamma}{\partial x}, \frac{\partial^2 \gamma}{\partial x^2}$ are cont in Q_1 and Q_2 (with one sided derivatives on the diagonal)

c) For ξ fixed γ (considered as a function of x) solves $L\gamma = 0$ on $Q \setminus \{x = \xi\}$

d) On $x = \xi$, $\frac{\partial \gamma}{\partial x}$ makes a jump of magnitude $\frac{1}{p}$, $\gamma_x(x+, x) - \gamma_x(x-, x) = \frac{1}{p(x)}$

Lemma Fundamental solutions exist

Proof Let $u(x,\xi)$ solve



$$Lu=0$$
$$u(\xi)=0$$
$$u'(\xi)=\frac{1}{p(\xi)}$$

and let
$$\gamma(x,\xi)=\begin{cases}0 & 0\le x\le \xi\le b\\ u(x,\xi) & a\le \xi\le x\le b.\end{cases}$$

This is not unique because
$$\tilde{\gamma}=\gamma+g(x)\,h(\xi) \quad \text{is also a solution}$$
if $Lg=0$ $\quad (g\in C^2(J))$ $\quad h\in C^0(J)$.

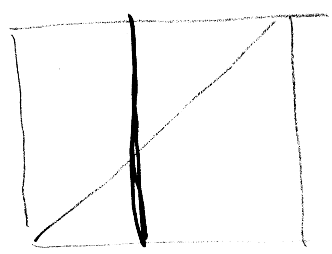
Theorem Consider $\begin{cases}Lu=g & \text{in } J\\ R_1u=R_2u\end{cases}$

and that the hypotheses on p, q, R_1, R_2 hold.

Then
$$v(x)=\int_a^b \gamma(x,\xi)\,g(\xi)\,d\xi \quad \text{is in } C^2(J)$$
and $Lv=g$ in J.

Proof
$$v(x)=\int_a^x \gamma(x,\xi)\,g(\xi)\,d\xi+\int_x^b \gamma(x,\xi)\,g(\xi)\,d\xi$$

$$v'(x)=\gamma(x,x)\,g(x)+\int_a^x \frac{\partial\gamma}{\partial x}(x,\xi)\,g(\xi)\,d\xi$$
$$-\gamma(x,x)\,g(x)+\int_x^b \frac{\partial\gamma}{\partial x}(x,\xi)\,g(\xi)\,d\xi$$
$$=\int_a^b \frac{\partial\gamma}{\partial x}(x,\xi)\,g(\xi)\,d\xi.$$



$$v''(x)=\frac{\partial\gamma}{\partial x}(x^+,x)\,g(x)-\frac{\partial\gamma}{\partial x}(\bar{x},x)\,g(x)$$
$$+\int_a^b \frac{\partial^2\gamma}{\partial x^2}(x,\xi)\,g(\xi)\,d\xi$$
$$=\int_a^b \frac{\partial^2\gamma}{\partial x^2}(x,\xi)\,g(\xi)\,d\xi+\frac{g(x)}{p(x)}.$$

$$Lv = pv'' + p'v' + qv$$

$$= p\int \frac{\partial^2 \gamma}{\partial x^2} g \, d\xi \quad + \quad p'\int \frac{\partial \gamma}{\partial x} g \, d\xi \quad + \quad q\int \gamma g \, d\xi$$

$$+ g(x)$$

$$= g \qquad \text{because} \qquad L\gamma = 0 .$$

■

<u>Def</u> Let $\begin{cases} Lu = 0 \\ R_1 u = R_2 u \end{cases}$ define a Sturmian boundary value problem. A <u>Green's function</u> is a function $\Gamma(x, \xi) : J \times J \to \mathbb{R}$ such that

a) $\Gamma(x, \xi)$ is a fundamental solution to $Lu = 0$

b) $R_1 \Gamma = R_2 \Gamma = 0$ for each $\xi \in J^\circ =]a, b[$

Assuming that $Lu = 0$ has only the trivial solution, the Green's function exists, and can be constructed as follows:

Let u_1, u_2 satisfy $Lu_1 = Lu_2 = 0$ and $R_1 u_1 = 0$, $R_2 u_2 = 0$. These can be obtained as

$$\begin{cases} Lu_1 = 0 \\ u_1(a) = \lambda \quad p(a) u_1'(a) = \mu \end{cases}$$

where $(\lambda, \mu) \neq (0,0)$ and $\alpha_1 \lambda + \beta_1 \mu = 0$ (so that $R_1 u_1 = 0$)

and similarly for u_2.

u_1, u_2 is a fundamental system, because if u_1 were proportional to u_2, then we would have $R_2 u_1 = R_1 u_2 = 0$, giving non-trivial solutions to $\begin{cases} Lu = 0 \\ R_1 u = R_2 u = 0 \end{cases}$

Let $\quad c = p(u_1 u_2' - u_1' u_2)$

Then c is a non-zero constant, because of Lagrange's identity.

Now let

$$\Gamma(x,\xi) = \frac{1}{c}\begin{cases} u_1(\xi)\,u_2(x) & \text{in } \quad a \le \xi \le x \le b \\ u_1(x)\,u_2(\xi) & \text{in } \quad a \le x \le \xi \le b \end{cases} \qquad (*)$$

Theorem If the homogeneous problem has only the trivial solution, then the Green's function exist and is unique.

Moreover, the solution to

$$\begin{cases} Lu = g \\ R_1 u = R_2 u = 0 \end{cases}$$

is given by

$$u(x) = \int_a^b \Gamma(x,\xi)\, g(\xi)\,d\xi\,.$$

Proof We need to prove uniqueness.

Assume there are two Green's functions, $\Gamma(x,\xi)$ and $\Gamma'(x,\xi)$

Let $\quad v(x) = \int_a^b \Gamma(x,\xi)\, g(\xi)\,d\xi, \qquad w(x) = \int_a^b \Gamma'(x,\xi)\, h(\xi)\,d\xi \qquad (**)$

where g and h are continuous.

Because $\quad R_i v = R_i w = 0 \quad (i=1,2)$

$$\int_a^b (vLw - wLv)\,dx = 0 \qquad \text{(Lagrange)}$$

Replace Lw by h and Lv by g and use $(**)$:

Then

$$\int_a^b \Big(\int_a^b \Gamma(x,\xi)\, g(\xi)\, d\xi\; h(x) - \int_a^b \Gamma'(x,\xi) h(\xi) d\xi\; g(x) \Big)\, dx$$

$$= \int_a^b \int_a^b \Big(\Gamma(x,\xi)\; g(\xi) h(x) - \Gamma'(x,\xi)\; h(\xi) g(x) \Big)\, dx\, d\xi$$

$$= \int_a^b \int_a^b \Big(\Gamma(x,\xi) - \Gamma'(\xi,x) \Big)\; h(x) g(\xi)\; dx\, d\xi .$$

Here we have assumed that it is ok to change the order of integration (Fubini's theorem).

But $\Gamma'(\xi,x) = \Gamma'(x,\xi)$ so

$$\int_a^b \int_a^b \big(\Gamma(x,\xi) - \Gamma'(x,\xi) \big)\; h(x)\; g(\xi)\; dx\, d\xi = 0$$

This holds for arbitrary functions $h(x)$ and $g(\xi)$, and therefore $\Gamma(x,\xi) = \Gamma'(x,\xi)$.

__Ex__ $\quad Lu = u'' \qquad R_1 u = u(0)\;, \quad R_2 u = u(1) \qquad (\,\mathcal{I} = [0,1]\,)$

Then $$\Gamma(x,\xi) = \begin{cases} \xi\,(x-1) & 0 \le \xi \le x \le 1 \\ x\,(\xi-1) & 0 \le x \le \xi \le 1 \end{cases}$$