Inner products and Hilbert spaces

Let V be a linear space.

An inner product is a function
$(\cdot,\cdot): V\times V \to \mathbb{R}$ such that if $v_1, v_2, v_3 \in V$
and $\alpha, \beta \in \mathbb{R}$, then

- $(v_1, \alpha v_2 + \beta v_3) = \alpha(v_1, v_2) + \beta(v_1, v_3)$
- $(v_1, v_2) = (v_2, v_1)$
- $(v_1, v_1) \geq 0$ and $(v_1, v_1) = 0 \Leftrightarrow v_1 = 0$

A scalar product naturally gives a norm,
$\|v\| = \sqrt{(v,v)}$.

If V is complete with respect to this norm, it is called a <u>Hilbert space</u>.

Let $T: V \to V$ be an operator.
T is said to be bounded if
$\|Tu\| \leq C\|u\|$ for some constant $C>0$ and every $u \in V$.

T is said to be compact if for each bounded sequence $\{v_n\}$, the sequence $\{Tv_n\}$ contains a bounded subsequence, i.e. there is $w \in V$ so that $Tv_{n_j} \to w$.

T is <u>self adjoint</u> if $(Tu, v) = (u, Tv)$ for all $u, v \in V$.

Example

$$\tilde{V} = C([0,1]) \qquad (u,v) = \int_0^1 u(t)v(t)\,dt$$

<u>not complete</u>

But $L^2([0,1])$ is defined as the "completion" of this space, much in the same way as $\mathbb{R}$ is the completion of $\mathbb{Q}$.

Let Tv be defined as

$$Tv(x) = \int_0^x v(t)\,dt.$$

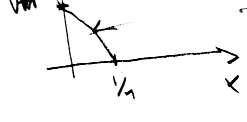
Then $\left(\int_0^x v(t)\,dt\right)^2 \le \left(\int_0^x |v(t)|\,dt\right)^2 \le$

$$\left(\int_0^1 |v(t)|\,dt\right)^2 \le \int_0^1 |v(t)|^2\,dt$$

$$\Rightarrow \|Tv\|^2 = \int_0^1 Tv(x)\,Tv(x)\,dx = \int_0^1 \left(\int_0^x v(t)\,dt\right)^2 dx \le \underbrace{\int_0^1 |v(x)|^2\,dx}_{=\|v\|^2}$$

So $\|T\| \le 1$.

On the other hand $D: v \longmapsto v'$ is an unbounded operator.

Let $v_n(x) =$  $\sqrt{n - n^2 x}$

$1/n$

Then $\|v_n\|^2 = \int_0^1 v_n(x)^2\,dx = \int_0^{1/n} (n - n^2 x)\,dx = \frac{1}{2}$

but $v_n'(x) = \begin{cases} -\dfrac{n^2}{2\sqrt{n - n^2 x}} & 0 < x < 1/n \\ 0 & x > 1/n \end{cases}$

$$\Rightarrow \|Dv_n\| = \int_0^{1/n} \frac{n^3}{4(n - n^2 x)}\,dx = \frac{n^2}{4}\int_0^{1/n} \frac{1}{1 - nx}\,dx = \infty \quad \text{(divergent)}$$

Lemma If T is a selfadjoint bounded operator then

$$\|T\| = \sup_{v \in V} \{ (Tv,v), \|v\|=1 \}$$

Proof Let $\beta = \sup_{v \in V} \{ (Tv,v), \|v\|=1 \}$

Then Cauchy's inequality implies that

$$|(Tv,v)| \leq \|Tv\| \|v\| \leq \|T\| \|v\|^2 .$$

Hence $\beta \leq \|T\|$.

Moreover, for any $u, v \in V$

$$(Tu+Tv, u+v) - (Tu-Tv, u-v)$$
$$= 2(Tu,v) + 2(Tv,u), \quad \text{and}$$

$$|(Tu+Tv, u+v) - (Tu-Tv, u-v)| =$$
$$= |(T(u+v), u+v) - (T(u-v), u-v)|$$
$$\leq \beta(\|u+v\|^2 + \|u-v\|^2) = 2\beta(\|v\|^2 + \|u\|^2).$$

Now take $u = \lambda h$ and $v = Th$, $\lambda = \|Th\|$, $\|h\|=1$

Then $2(Tu,v) + 2(Tv,u) =$
$$2\lambda(Th,Th) + 2\lambda(TTh,h) = 4\lambda^3$$

So $4\lambda^3 \leq 2\beta(\|\lambda h\|^2 + \|Th\|^2) = 4\beta\lambda^2$

$\Rightarrow \quad \lambda = \|Th\| \leq \beta \quad \Rightarrow \quad \|T\| \leq \beta$ because h was arbitrary.

Def Let V be an inner product space.

A sequence $\{u_n\}$ is called an orthonormal system if $(u_n, u_m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$

Let $f \in H$ and $c_i = (f, u_i)$. Then

$\sum c_i u_i$ is called the Fourier series generated by f.

Consider a finite sum $\sum_{i \in J} \_\_$, where J is a finite set of integers.

Then

$$(f - \sum_{i \in J} d_i u_i, f - \sum_{i \in J} d_i u_i) =$$

$$= (f, f) - \sum_{i \in J} d_i (u_i, f) - \sum_{i \in J} d_i (f, u_i) + \sum_{i,j} d_i d_j$$

$$= (f, f) + \sum_{i \in J} |d_i - c_i|^2 - \sum |c_i|^2$$

$\Rightarrow$ best approximation if $d_i = c_i$

Let $s_n = \sum_{i=0}^{n} c_i u_i$

Then $\|f - s_n\|^2 = \|f^2\| - \sum_{i=0}^{n} |c_i|^2$

$$\Rightarrow \quad \sum_{i=0}^{\infty} |c_i|^2 \leq \|f\|^2 \qquad \text{(Bessels inequality).}$$

If there is equality, then the orthonormal system is said to be complete.

<u>Def</u> Let T be a linear operator $H \to H$ where H is a (real) Hilbert space.

T is said to be compact if for any bounded sequence $v_j \in H$, $(\|v_j\| \le C \quad j=1,\dots\infty)$ the sequence $Tv_j \in H$ contains a convergent subsequence.

<u>Theorem</u> If T is compact and selfadjoint then there is an eigenvalue $\mu_0 \in \mathbb{R}$, and an eigenvector u_0, so that

$$|\mu_0| = \|T\|, \quad \|u_0\|=1, \quad Tu_0 = \mu_0 u_0.$$

<u>Proof</u> Take a sequence $\phi_n \in H$, $\|\phi_n\|=1$ and such that $|(T\phi_n, \phi_n)| \to \|T\| \underset{\uparrow \text{def}}{=} \mu$

Because T is compact, we may assume

$T\phi_n \to \mu u$ where $\|u\|=1$.

Then $\|T\phi_n - \mu\phi_n\|^2 = \|T\phi_n\|^2 - 2\mu(T\phi_n, \phi_n) + \mu^2\|\phi_n\|^2$

$$= 2\mu^2 - 2\mu(T\phi_n, \phi_n) \to 0$$

$\Rightarrow$ $T\phi_n = \mu\phi_n + \varepsilon_n$ where $\varepsilon_n \in H$ and $\|\varepsilon_n\| \to 0$

But $T\phi_n \to \mu u$ so

$$\mu u - \mu\phi_n \to 0 \quad \Rightarrow \quad \phi_n \to u$$

and because T is linear and bounded,

$T\phi_n \to Tu$.

So $Tu = \mu u$.

Theorem Let H be an infinite dimensional inner product space, and let T be a linear, selfadjoint, compact operator.

Then the eigenvalue problem $Tu = \mu u$ has countably many eigenvalues (real), $\mu_0, \mu_1, \ldots, \mu_n, \mu_{n+1} \ldots$ such that $|\mu_0| \geq |\mu_1| \geq |\mu_2| \geq \ldots \to 0$ when $n \to \infty$

The corresponding eigenvectors $u_0, u_1, \ldots u_n$ satisfy $(u_m, u_n) = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m \neq n. \end{cases}$

(i.e. eigenvectors are orthogonal)

Let $H_n = \{ f \in H : (f, u_i) = 0, \ i = 0, 1, 2, \ldots, n-1 \}$

Then $|\mu_n| = \sup\limits_{f \in H_n, \|f\|=1} \|Tf\|$, $(Tu_n, u_n) = \mu_n \|u_n\|^2$

For $f \in H$, $Tf = \sum\limits_{i=0}^{\infty} \mu_i d_i u_i$ where $d_i = (f, u_i)$

Compare Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix which is symmetric. Then there is a base of eigenvectors $v_1 \ldots v_n$ so that $Av_j = \lambda_j v_j$ and so that for every $w \in \mathbb{R}^n$,

$$Aw = \sum_{i=1}^{n} \frac{w \cdot v_j}{|v_j|} \lambda_j v_j$$

because $w = \sum\limits_{j=1}^{n} \frac{w \cdot v_j}{|v_j|^2} v_j$.

Consider now Sturm-Liouville eigenvalue problem:

- $$\begin{cases} Lu + \lambda r u = 0 & \text{in } J = [x_1, x_2] \\ R_1 u = R_2 u = 0 \end{cases} \qquad r \in C(J),\ r > 0$$

$Lu = (pu')' + qu$, $R_i u$ as before.

(assume $\lambda = 0$ is not a solution, because we can replace q by $q + \lambda_* r$, where λ_* is not an eigenvalue).

Then $\begin{cases} Lu = -\lambda r u \\ R_1 u = R_2 u = 0 \end{cases}$

$$\Rightarrow \quad u(x) = -\lambda \int_{x_1}^{x_2} \Gamma(x,\xi)\, r(\xi)\, u(\xi)\, d\xi$$

where Γ is the Green's function.

Let $Tf = -\int \Gamma(x,\xi) r(\xi) f(\xi) d\xi$.

Then we obtain a new eigenvalue problem

$$Tu = -\frac{1}{\lambda} u.$$

Define $(f,g)_r = \int f(x) g(x) r(x) dx$.

- $Tf = -\int \Gamma(x,\xi)\, r(\xi) f(\xi)\, d\xi$ is linear
- $(Tf, g)_r = \int Tf(x)\, g(x)\, r(x)\, dx =$

$$= -\iint r(x) g(x)\, \Gamma(x,\xi)\, f(\xi)\, r(\xi)\, dx\, d\xi, \quad \text{symmetric}$$

- T is compact.

(Arzelà-Ascoli).