1 Find the general solution of each of the following differential equations:

a $xyy' = y - 1$

b $(1 + x^2)\, dy + (1 + y^2)\, dx = 0$

2 For each of the following differential equations, find the integral curve that passes through the given point:

a $3 \cos 3x \cos 2y\, dx - 2 \sin 3x \sin 2y\, dy = 0$, $(\pi/12, \pi/8)$

b $y' = e^{3x-2y}$, $(0, 0)$

3 A saddle without a saddle-horn (pommel) has the shape of the surface $z = y^2 - x^2$. It is lying outdoors in a rainstorm. Find the paths along which raindrops will run down the saddle. Draw a sketch and use it to convince yourself that your answer is reasonable.

4 Find the differential equation of each of the following one-parameter families of curves:

a all circles with centers on the line $y = x$ and tangent to both axes;

b all lines tangent to the parabola $x^2 = 4y$ (*hint*: the slope of the tangent line at $(2a, a^2)$ is a);

5 Suppose that two chemical substances in solution react together to form a compound. If the reaction occurs by means of the collision and interaction of the molecules of the substances, then we expect the rate of formation of the compound to be proportional to the number of collisions per unit time, which in turn is jointly proportional to the amounts of the substances that are untransformed. A chemical reaction that proceeds in this manner is called a *second order reaction*, and this law of reaction is often referred to as the *law of mass action*. Consider a second order reaction in which x grams of the compound contain ax grams of the first substance and bx grams of the second, where $a + b = 1$. If there are aA grams of the first substance present initially, and bB grams of the second, and if $x = 0$ when $t = 0$, find x as a function of the time t.[12]

6 Consider a column of air of cross-sectional area 1 square inch extending from sea level up to "infinity." The atmospheric pressure p at an altitude h above sea level is the weight of the air in this column above the altitude h. Assuming that the density of the air is proportional to the pressure, show that p satisfies the differential equation

$$\frac{dp}{dh} = -cp, \qquad c > 0,$$

(a) Show that the length of the part of the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$ that lies in the first quadrant is

$$\int_0^a \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\, dx,$$

where e is the eccentricity.

(b) Use the change of variable $x = a \sin \phi$ to transform the integral in (a) into

$$a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \phi}\, d\phi = aE(e, \pi/2),$$

7 The clepsydra, or ancient water clock, was a bowl from which water was allowed to escape through a small hole in the bottom. It was often used in Greek and Roman courts to time the speeches of lawyers, in order to keep them from talking too much. Find the shape it should have if the water level is to fall at a constant rate.

8 According to *Torricelli's law,* water in an open tank will flow out through a small hole in the bottom with the speed it would acquire in falling freely from the water level to the hole. A hemispherical bowl of radius R is initially full of water, and a small circular hole of radius r is punched in the bottom at time $t = 0$. How long will the bowl take to empty itself?

9 Four bugs sit at the corners of a square table of side a. At the same instant they all begin to walk with the same speed, each moving steadily toward the bug on its right. If a polar coordinate system is established on the table, with the origin at the center and the polar axis along a diagonal, find the path of the bug that starts on the polar axis and the total distance it walks before all bugs meet at the center.

10 Verify that the following equations are homogeneous, and solve them:

a $x^2y' = 3(x^2 + y^2)\tan^{-1}\frac{y}{x} + xy$; b $(x^2 - 2y^2)\,dx + xy\,dy = 0$;

11 Show that the substitution $z = ax + by + c$ changes

$$y' = f(ax + by + c)$$

into an equation with separable variables, and apply this method to solve the following equations:
(a) $y' = (x + y)^2$; (b) $y' = \sin^2(x - y + 1)$.

13

a If $ae \neq bd$, show that constants h and k can be chosen in such a way that the substitutions $x = z - h$, $y = w - k$ reduce

$$\frac{dy}{dx} = F\left(\frac{ax + by + c}{dx + ey + f}\right)$$

to a homogeneous equation.

b If $ae = bd$, discover a substitution that reduces the equation in (a) to one in which the variables are separable.

Let $y' = f(x,y)$ be a homogeneous differential equation, and prove the following geometric fact about its family of integral curves: If the xy-plane is stretched from (or contracted toward) the origin in such a way that each point (x,y) is moved to a new point (x_1,y_1) which is k times its original distance from the origin, with its direction from the origin unchanged, then every integral curve C is carried into an integral curve C_1. *Hint:* $x_1 = kx$ and $y_1 = ky$.

14 The equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

which is known as *Bernoulli's equation,* is linear when $n = 0$ or 1. Show that it can be reduced to a linear equation for any other value of n by the change of variable $z = y^{1-n}$, and apply this method to solve the following equations:

(a) $xy' + y = x^4y^3$;

(b) $xy^2y' + y^3 = x \cos x$;

(c) $x\,dy + y\,dx = xy^2\,dx$.

15 We know that the general solution of a first order linear equation is a family of curves of the form

$$y = cf(x) + g(x).$$

Show, conversely, that the differential equation of any such family is linear.

16 A natural extension of the first order linear equation

$$y' = p(x) + q(x)y$$

is the *Riccati equation*

$$y' = p(x) + q(x)y + r(x)y^2.$$

In general, this equation cannot be solved by elementary methods. However, if a particular solution $y_1(x)$ is known, then the general solution has the form

$$y(x) = y_1(x) + z(x)$$

where $z(x)$ is the general solution of the Bernoulli equation

$$z' - (q + 2ry_1)z = rz^2.$$

Prove this, and find the general solution of the equation

$$y' = \frac{y}{x} + x^3y^2 - x^5,$$

which has $y_1(x) = x$ as an obvious particular solution.

17

Einstein's special theory of relativity asserts that the mass m of a particle moving with velocity v is given by the formula

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \qquad (*)$$

where c is the velocity of light and m_0 is the rest mass.

(a) If the particle starts from rest in empty space and moves for a long time under the influence of a constant gravitational field, find v as a function of time by taking $w = -v$, and show that $v \to c$ as $t \to \infty$.[7]

(b) Let $M = m - m_0$ be the increase in the mass of the particle. If the corresponding increase E in its energy is taken to be the work done on it by the prevailing force F, so that

$$E = \int_0^v F\,dx = \int_0^v \frac{d}{dt}(mv)\,dx = \int_0^v v\,d(mv),$$

verify that

$$E = Mc^2. \qquad (**)$$

(c) Deduce $(*)$ from $(**)$.