Former Examination 2

1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\underline{y}_0 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Compute the solution of
$$\underline{y}'(t) = A \cdot \underline{y}(t),\ \underline{y}(0) = \underline{y}_0.$$

2. Compute the solution of

   (a)
   $$x'(t) = 4\,t^3\,x(t)^2.$$

   (b)
   $$x'(t) = e^{-t}\,x(t)^2.$$

3. Solve with Picard iteration the problem
$$x'(t) = 5\,x(t), \quad x(0) = 1.$$

4. Solve the initial value problem
$$x_1'(t) = x_1(t) + 2\,x_2(t) - 2\,e^t,$$
$$x_2'(t) = 4\,x_1(t) + 3\,x_2(t) - 10\,e^t,$$
$$x_1(0) = 6, \quad x_2(0) = 0.$$

5. Which of the following boundary value problems can be solved for all continuous functions f?

   (a)
   $$y'' + y = f, \quad y(0) = y(\pi/2) = 0.$$

   (b)
   $$y'' + y = f, \quad y(0) = y(\pi) = 0.$$

6. (a) Determine eigenvalues and eigenfunctions of
   $$y'' + y = \lambda\,y, \quad y'(0) = y'(1) = 0.$$

   (b) Solve the equation
   $$y''(x) + y(x) = \cos(4\,\pi\,x), \quad y'(0) = y'(1) = 0.$$

7. Let $A : (a, b) \to \mathbb{R}^{n\times n}$ be continuous. Show that $\underline{y}'(t) = A \cdot \underline{y}(t)$ has n solutions $\underline{y}_1, \underline{y}_2, \ldots \underline{y}_n$ such that for all $t \in (a, b)$ the vectors

$$\underline{y}_1(t),\ \underline{y}_2(t), \ldots, \underline{y}_n(t)$$

are linearly independent.

8. Let $\mathcal{L}$ be a symmetric operator. Show that eigenvectors corresponding to different eigenvalues are orthogonal to each other.

Note that the exercises in the examinations in 2003 can be different from the ones given here and also might be similar. Thus in any way one should learn a lot and, in particular, solve many exercises, before the examination.