Former Examination 3

1. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $\underline{y}_0 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Determine the solution to

$$\underline{y}'(t) = A \cdot \underline{y}(t),\ \underline{y}(0) = \underline{y}_0.$$

2. Determine all equilibrium points of the following differential equation, give a diffential equation for the trajectories of the system, compute the solutions to the diffential equation for the trajectories and scetch the phase portrait.

$$\begin{aligned} x'(t) &= y(t)(x(t)^2 + 1), \\ y'(t) &= -x(t)(x(t)^2 + 1). \end{aligned}$$

3. a) Determine the eigenvalues and eigenfunctions of

$$y'' + y = \lambda\, y, \quad y'(0) = y'(1) = 0.$$

b) Determine the solution to

$$y''(x) + y(x) = \cos(4\,\pi\, x), \quad y'(0) = y'(1) = 0.$$

4. Determine all solutions to

$$y'(t) = (1 + e^t)y(t)^2.$$

5. Determine the general solution to

$$\begin{aligned} y_1'(t) &= -4y_1(t) - 6y_2(t) + 3\sin(t), \\ y_2'(t) &= y_1(t) + y_2(t) + 2\sin(t). \end{aligned}$$

6. For which a is the system below asymptotically stable?

$$\begin{aligned} x_1'(t) &= 2x_1(t) + ax_2(t), \\ x_2'(t) &= x_1(t) - 4x_2(t) \end{aligned}$$

7. Formulate and prove Banach's fixpoint theorem.

8. Prove the following: Let G be an invertible symmetric integral operator in $C[\alpha, \beta]$ with a continuous kernel $g(x, y)$ and suppose that the range of G is dense in $C[a, b]$. Then $C[\alpha, \beta]$ has an orthonormal basis $(e_k)_{k=1}^{\infty}$ of eigenfunctions of G and the corresponding eigenvalues converge to 0 as $k \longrightarrow \infty$.