MATEMATIK
Tentamen 2005–08–26 kl. 08.30–13.30

MAN460/TMA013 Ordinary differential equations
PARTIAL SOLUTIONS

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3 Solve the following first order equations:
a) $y' + 2xy = e^{-x^2}$ b) $y' = 1 + \frac{y}{x} - \left(\frac{y}{x}\right)^2$ (5p)

Solution: a) This is a linear first order differential equation that can be solved with the integrating factor e^{x^2}:

$$\frac{d}{dx}\left(y(x)e^{x^2}\right) = e^{x^2}e^{-x^2} = 1\,,$$

and hence

$$y(x)e^{x^2} = x + C \qquad \Leftrightarrow \qquad y(x) = (x + C)e^{-x^2}$$

Solution: b) Let $u = y/x$. Then $y' = u + xu'$, and then $u + xu' = 1 + u - u^2$, which is separable:

$$\frac{u'}{1-u^2} = \frac{1}{x} \qquad \Leftrightarrow \qquad \frac{u'}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right) = \frac{1}{x}\,,$$

which can be integrated to

$$\log\left|\frac{1+u}{1-u}\right| = 2\log(x) + C \qquad \text{or} \qquad \frac{1+u}{1-u} = C'x^2$$

from which one can solve for $u(x)$, and then go back to y:

$$y(x) = x\frac{C'x^2 - 1}{C'x^2 + 1}$$

5 Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Compute a) $\sin(A)$; b) $\exp(A)$.

Solution: One can use the Cayley-Hamilton theorem. The characteristic equation is $(3-\lambda)^2 - 1 = 0$, i.e. $(\lambda - 2)(\lambda - 4) = 0$. We write

$$\sin(x) = p_1(x)(x-2)(x-4) + a_1 + b_1 x$$
$$\exp(x) = p_2(x)(x-2)(x-4) + a_2 + b_2 x$$

Setting $x = 2$ and $x = 4$ gives

$$a_1 = 2\sin 2 - \sin 4 \qquad b_1 = \frac{1}{2}(\sin 4 - \sin 2)$$
$$a_1 = 2\exp 2 - \exp 4 \qquad b_1 = \frac{1}{2}(\exp 4 - \exp 2)$$

It follows that

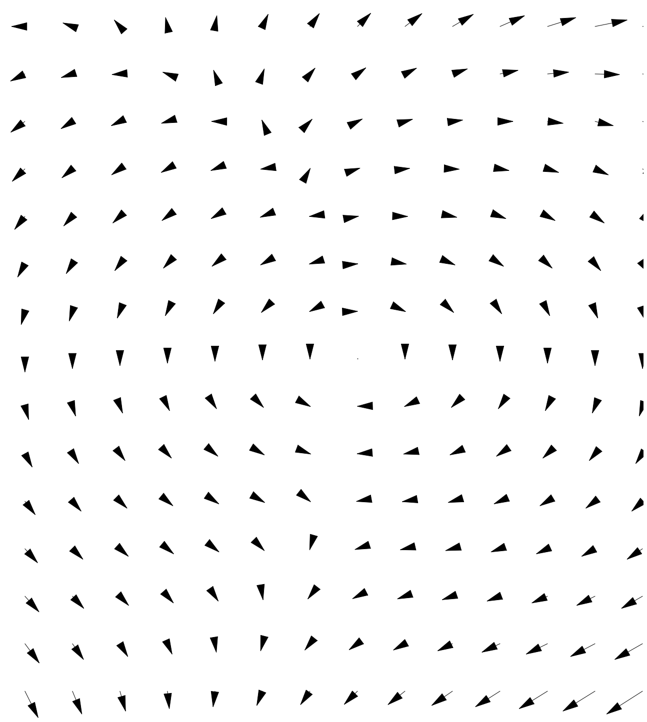
$$\sin A = a_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b_1\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\,,$$
$$\exp A = a_2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b_2\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

which of course should be simplified. (4p)

6 Sketch the phase portrait for the following system:

$$\begin{cases} \dot{x} = 2xy + x^3 \\ \dot{y} = -x^2 + y^5 \end{cases}$$

Solution:(generated with Mathematica ...)



(2p)

7 Solve the equation $xy^2y' + y^3 = x\cos(x)$.
Hint: try a substitution of the form $z = x^k$ for a suitably chosen integer k.

Soln: Set $u = y^3$, and obtain $u' = 3y^2y'$, which gives

$$u' + \frac{3}{x}u = 3\cos(x)\,,$$

which can be solved by using the integrating factor x^3:

$$\frac{d}{dx}\left(x^3y^3\right) \quad = \quad 3x^2\cos(x)\,,$$

which can be integrated to $x^3y^3 = 2x\cos x + +(x^2-2)\sin x + C$, from which one easily finds $y(x)$.

8 Show that any equation of the form $P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$ can be made self-adjoint by multiplying with the function $\frac{1}{P}\exp(\int (Q/P)\,dx)$.

Soln: This is just a matter of carrying out the differentiation, simplifying, and noticing that the odd-order terms dissapear.