Natural Deduction.

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Natural deduction was invented by G. Gentzen in 1934. The idea was to have a system of derivation rules that as closely as possible reflect the logical steps in an informal rigorous proof. For each connective there is an introduction rule (except conjunction, which has two) which can be seen as a definition of which conditions must be satisfied for the proposition to be true.

A derivation is a tree where the nodes are the rules and the leafs are the assumptions of the derivation. The root of the tree is the *conclusion* of the derivation. An assumption may be *discharged* by a rule in the derivation; that an assumption A is discharged in a derivation is denoted by $[A]$.

1 Propositional logic

If we have a derivation of B from the assumption A, we may derive $A \rightarrow B$ and discharge the assumption A:

$\rightarrow I$

$$\frac{\begin{array}{c}[A]\\ B\end{array}}{A \rightarrow B}$$

Given derivations of $A \rightarrow B$ and A, we can conclude B:

$\rightarrow E$

$$\frac{A \rightarrow B \qquad A}{B}$$

For conjunction, the introduction rule tells us that if we have derivations of A and B, then we can conclude $A \wedge B$:

$\wedge I$

$$\frac{A \qquad B}{A \wedge B}$$

There are two elimination rules for conjunction:

$\wedge E1$ $$\frac{A \wedge B}{A}$$

$\wedge E2$ $$\frac{A \wedge B}{B}$$

We can infer a disjunction if we know one of the disjuncts. Hence there are two introduction rules for the disjunction:

$\vee I1$ $$\frac{A}{A \vee B}$$

$\vee I2$ $$\frac{B}{A \vee B}$$

The elimination rule for disjunction is slightly more complicated to formulate. We must make use of the fact that we know $A \vee B$ but not necessarily which of A and B it is that holds.

$\vee E$ $$\frac{A \vee B \quad \begin{array}{c}[A]\\ C\end{array} \quad \begin{array}{c}[B]\\ C\end{array}}{C}$$

For constructive propositional logic, there only remains one rule, the absurdity elimination:

$\perp E$ $$\frac{\perp}{C}$$

1.1 Some examples

The simplest possible derivation just consists of an assumption:

$$A$$

which shows that $A \vdash A$. Next is a derivation of $A \wedge B \to A$ in the empty context, i.e. $\vdash A \wedge B \to A$.

$$\dfrac{\dfrac{[A \wedge B]^1}{A}\,\wedge E1}{A \wedge B \to A}\to I_1$$

In this derivation we have given the name of the rule used and also put number on the assumption so that we can refer to it in the rule it is discharged. The tree of this derivation is

The next example shows that conjunction is commutative.

$$\dfrac{\dfrac{\dfrac{[A\wedge B]^1}{A}\,\wedge E1 \qquad \dfrac{[A\wedge B]^1}{B}\,\wedge E2}{B\wedge A}\,\wedge I}{A\wedge B\to B\wedge A}\,\to I_1$$

Note that in this derivation the formula $A \wedge B$ occurs as an assumption in several leafs which all are discharged by the same rule. It is not necessary that a rule which may discharge an assumption actually discharges any assumption; this is used in the following derivation.

$$\dfrac{\dfrac{[A]^1}{B\to A}\,\to I}{A\to(B\to A)}\,\to I_1$$

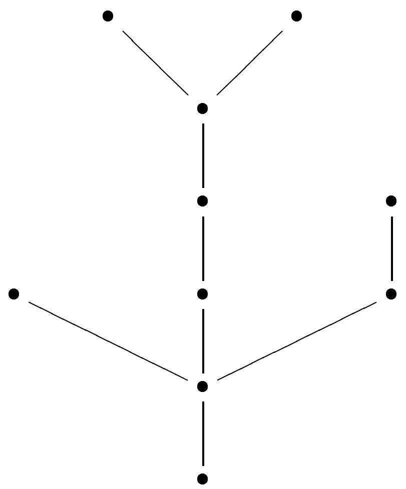
Since $\neg A \;=\; A \to \bot$, the next derivation proves $\vdash A \to \neg\neg A$.

$$\dfrac{\dfrac{\dfrac{[A\to\bot]^2 \quad [A]^1}{\bot}\,\to E}{(A\to\bot)\to\bot}\,\to I_2}{A\to((A\to\bot)\to\bot)}\,\to I_1$$

Finally, an example in which $\vee E$ is used.

$$\dfrac{\dfrac{[\neg A\vee B]^1 \qquad \dfrac{\dfrac{\dfrac{[A\to\bot]^3 \quad [A]^2}{\bot}\,\to E}{B}\,\bot E}{A\to B}\,\to I \qquad \dfrac{[B]^4}{A\to B}\,\to I_4}{A\to B}\,\vee E_{3,4}}{(\neg A\vee B)\to(A\to B)}\,\to I_1$$

The tree is



1.2 Classical logic

The rules we have given so far can all be justified constructively, but that does not hold for *reductio ad absurdum* which introduces the possibility of making proof by contradiction.

RAA

$$\dfrac{\begin{array}{c}[\neg A]\\ \bot\end{array}}{A}$$

With RAA we can prove that the double negation of a proposition implies the proposition:

$$\dfrac{\dfrac{\dfrac{[A \to \bot]^2 \quad [(A \to \bot) \to \bot]^1}{\bot} \to E}{A}\, RAA_2}{((A \to \bot) \to \bot) \to A} \to I_1$$

We can also prove the law of the excluded middle:

$$\dfrac{\dfrac{[\neg(A\vee\neg A)]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg(A\vee\neg A)]^1 \quad \dfrac{[A]^2}{A\vee\neg A}\,\vee I1}{\bot}\,\to E}{\neg A}\,\to I_2}{A\vee\neg A}\,\vee I2}{\bot}\,\to E}{A\vee\neg A}\,RAA_1$$

1.3 Exercises

1. Derive the following formulas

   (a) $(A \to (B \to C)) \to (A \wedge B \to C)$

   (b) $(A \wedge B \to C) \to (A \to (B \to C))$

   (c) $(A \to B) \to ((B \to C) \to (A \to C))$

   (d) $A \wedge B \to \neg(A \to \neg B)$

   (e) $\neg(A \to \neg B) \to A \wedge B$ (RAA is needed.)

   (f) $\neg\bot$

   (g) $\neg\neg\neg A \to \neg A$ (This can be derived without RAA.)

2. Show

   (a) $A \to (B \to C) \vdash B \to (A \to C)$

   (b) $\neg A \vdash A \to B$

   (c) If $\vdash B$ then $\vdash A \to B$.

   (d) $A \to (B \to C), A \vee C, \neg B \to \neg A \vdash C$

2 Predicate logic

The rules for predicate logic are those of propositional logic plus introduction and elimination rules for the quantifiers.

You may conclude $\forall x A(x)$ if you have derived $A(z)$ for an arbitrary z:

$\forall I$
$$\frac{A(z)}{\forall x A(x)}$$

That z is arbitrary means here that z must not occur in $\forall x A(x)$ nor in any assumption on which the derivation of $A(x)$ depends. The variable z is called the *eigenvariable* of the application of the rule.

The elimination rule says that if you have derived $\forall x A(x)$ you may conclude $A(t)$ for any term t.

$\forall E$
$$\frac{\forall x A(x)}{A(t)}$$

As usual, no free variables in t must become bound when we substitute t for x in $A(x)$.

An existential proposition $\exists x A(x)$ may be concluded if we have a witness t of it.

$\exists I$
$$\frac{A(t)}{\exists x A(x)}$$

The elimination rule for the existential quantifier is a little bit more complicated to formulate (in a similar way as $\vee$-elimination).

$\exists E$
$$\frac{\exists x A(x) \qquad \begin{array}{c}[A(z)]\\ C\end{array}}{C}$$

The intuition behind the rule is the following. We have a derivation of $\exists x A(x)$, which however does not necessarily mean that we have a witness of the proposition. So instead we use the derivation of $\exists x A(x)$ to conclude a proposition C provided we can derive C from $A(z)$ where z is arbitrary. That z is arbitrary means here that z must not occur in $\exists x A(x)$ and C, nor in any other assumption than $A(z)$ which the derivation of the upper occurrence of C depends on. As for $\forall$-introduction, z is called the eigenvariable of the rule.

2.1 Some examples

In the first example, the variable x must not occur free in the formula A. And the eigenvariable z must be chosen so that it occurs neither in A, nor in $(\forall x)B(x)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A \to B(x))]^1}{A \to B(z)}\,\forall E \qquad [A]^2}{B(z)}\,\to E}{\forall x B(x)}\,\forall I}{A \to \forall x B(x)}\,\to I_2}{\forall x(A \to B(x)) \to (A \to \forall x B(x))}\,\to I_1$$

Note that the restrictions of $\forall I$ are satisfied. The implication also holds in the other direction (exercise!).

Already to show

$$\exists x(A(x) \wedge B(x)) \vdash \exists x A(x)$$

both $\exists I$ and $\exists E$ are needed:

$$\dfrac{\exists x(A(x) \wedge B(x)) \qquad \dfrac{\dfrac{[A(z) \wedge B(z)]^1}{A(z)}\wedge E1}{\exists x A(x)}\exists I}{\exists x A(x)}\exists E_1$$

2.2 Exercises

1. Derive the following formulas.

   (a) $(A \to \forall x B(x)) \leftrightarrow \forall x(A \to B(x))$ where x does not occur free in A.

   (b) $\exists x(A(x) \vee B(x)) \leftrightarrow (\exists x A(x) \vee \exists x B(x))$

   (c) $\neg \exists x A(x) \leftrightarrow \forall x \neg A(x)$

   (d) $\forall x(A(x) \vee B) \leftrightarrow \forall x A(x) \vee B$ where x does not occur free in B.

   (e) $\exists x(A(x) \to B) \leftrightarrow (\forall x A(x) \to B)$ where x does not occur free in B.

   (f) $\exists x(A \to B(x)) \leftrightarrow (A \to \exists x B(x))$ where x does not occur free in A.

   (g) $\forall x(A(x) \to B) \leftrightarrow (\exists x A(x) \to B)$ where x does not occur free in B.