GIBBS PHENOMENA

Gibbs phenomena is well illustrated by the function

$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ 0 & -\pi < x < 0 \end{cases}$$

The Fourier coefficients of the function can be evaluated by the definition:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt = \frac{1}{2\pi}\int_{0}^{\pi} e^{-int}dt = \frac{1}{2\pi}\frac{1}{-in}((-1)^n - 1) = \begin{cases} 0 & n \neq 0 \text{ is even;} \\ \frac{1}{i\pi n} & n \text{ is odd.;} \\ \frac{1}{2} & n = 0. \end{cases}$$

Let us now estimate $\sup\{S_N(f)\}$. In order to estimate the sup from bellow, it is enough to evaluate the sum at some point.

We see that $S_{2N+1}(f) = \frac{1}{2} + \sum\limits_{n=0}^{N}(\frac{1}{i\pi(2n+1)}e^{i(2n+1)x} - \frac{1}{i\pi(2n+1)}e^{-i(2n+1)x})$ is a smooth real valued function, so it likely attains its maximum at an extreme point. We can take the derivative even in the complex form (it still will be real as the sum), and we see that

$$S'_{2N+1} = \frac{1}{\pi}\sum_{n=-(N+1)}^{N} e^{i(2n+1)x} = \frac{1}{\pi}e^{-i(2N+1)x}\frac{1-e^{i4Nx}}{1-e^{i2x}}.$$

So, the likely point to give a good estimate is $x = \frac{\pi}{2N}$.

Now to evaluate $S_{2N+1}(f)(\frac{\pi}{2N})$ we observe, that

$$S_{2N+1}(f) = \frac{1}{2} + \frac{1}{\pi}Im(\sum_{n=0}^{N}\frac{1}{2n+1}e^{i(2n+1)x}) = \frac{1}{2} + \frac{2}{\pi}Im(\sum_{n=0}^{N}\frac{1}{2n+1}z^{(2n+1)}),$$

where $z = e^{ix}$. In order to evaluate $\phi(z) = \sum\limits_{n=0}^{N}\frac{1}{2n+1}z^{(2n+1)}$, we observe that $\phi'(z) = \sum\limits_{n=0}^{N} z^{2n} = \frac{1-z^{2(N+1)}}{1-z^2}$, and as $\phi(0) = 0$, we can reconstruct $\phi(z) = \int\limits_0^z \frac{1-\tau^{2(N+1)}}{1-\tau^2}d\tau$. (Those, who didn't got to a course in Complex Analysis have to trust that for a polynomial this formula is valid even for the complex numbers.)

As the result of the integration doesn't depend from the path of integration, it will be convenient to choose the path to $z = e^{i\frac{\pi}{2N}}$, as $\tau = zt$, where t runs from 0 to 1. This way we have

$$\phi(z) = e^{i\frac{\pi}{2N}}\int_0^1 \frac{1+e^{i\frac{\pi}{N}}t^{2(N+1)}}{1-e^{i\frac{\pi}{N}}t^2}dt.$$

Fortunately, we are interested only in $Im(\phi)$, which is

$$Im(\phi(z)) = \frac{1}{2i}(e^{i\frac{\pi}{2N}}\int_0^1 \frac{1+e^{i\frac{\pi}{N}}t^{2(N+1)}}{1-e^{i\frac{\pi}{N}}t^2}dt - e^{-i\frac{\pi}{2N}}\int_0^1 \frac{1+e^{-i\frac{\pi}{N}}t^{2(N+1)}}{1-e^{-i\frac{\pi}{N}}t^2}dt) =$$

$$\frac{1}{2i}\int_0^1 \frac{e^{i\frac{\pi}{2N}}+e^{i\frac{3\pi}{2N}}t^{2(N+1)}-e^{-i\frac{\pi}{2N}}t^2-e^{i\frac{\pi}{2N}}t^{2(N+2)}-e^{-i\frac{\pi}{2N}}-e^{-i\frac{3\pi}{2N}}t^{2(N+1)}+e^{i\frac{\pi}{2N}}t^2+e^{-i\frac{\pi}{2N}}t^{2(N+2)}}{1+t^4-2\cos(\frac{\pi}{N})t^2}dt =$$

$$\int_0^1 \frac{\sin(\frac{\pi}{2N})(1+t^2-t^{2(N+2)})+\sin(\frac{3\pi}{2N})t^{2(N+1)}}{(\cos(\frac{\pi}{N})-t^2)^2+\sin^2(\frac{\pi}{N})}dt \geq$$

(The estimate of this last integral is rather messy, which is the reason the Gibbs phenomena is normally not proven in the course.)

$$\int_0^1 \frac{\sin(\frac{\pi}{2N})(1+t^2-t^{2(N+2)})+\sin(\frac{3\pi}{2N})t^{2(N+1)}}{(\cos(\frac{\pi}{N})-t^2)^2+\sin^2(\frac{\pi}{N})}tdt =$$

$$\frac{1}{2}\int_0^1 \frac{\sin(\frac{\pi}{2N})(1+t-t^{(N+2)})+\sin(\frac{3\pi}{2N})t^{(N+1)}}{(\cos(\frac{\pi}{N})-t)^2+\sin^2(\frac{\pi}{N})}dt = \frac{1}{2}I_N.$$

We need only the asymptotic (in N) behavior of the integral, and so we aim only to evaluate its limit, when $N\to\infty$.

$$I_N = \int_0^{\cos(\frac{\pi}{N})} \frac{\sin(\frac{\pi}{2N})(1+t-t^{(N+2)})+\sin(\frac{3\pi}{2N})t^{(N+1)}}{(\cos(\frac{\pi}{N})-t)^2+\sin^2(\frac{\pi}{N})}dt +$$

$$\int_{\cos(\frac{\pi}{N})}^1 \frac{\sin(\frac{\pi}{2N})(1+t-t^{(N+2)})+\sin(\frac{3\pi}{2N})t^{(N+1)}}{(\cos(\frac{\pi}{N})-t)^2+\sin^2(\frac{\pi}{N})}dt = I_N^* + I_N^{**}.$$

The second integral is over the interval of length $O(1-\cos(\frac{\pi}{N})) = O(\frac{1}{N^2})$, while the function is $O(\frac{1}{N})$, which means $I_N^{**} = O(\frac{1}{N})$. This means $\lim_{N\to\infty} I_N = \lim_{N\to\infty} I_N^*$.

The first observation is that, while $\sin(x)\sim x$,

$$\int_{-\infty}^{\infty} \frac{1}{(a-t)^2+x^2}dt = \frac{1}{x}[\arctan(t)]_{-\infty}^{\infty} = \frac{\pi}{x} = O(\frac{1}{x}),$$

so the integral is finite, and we can disregard all errors of greater order. In particular we can change $\sin(\frac{\pi}{N})$ with $\frac{\pi}{N}$, etc.

The new integral is similar to I_N^*,

$$\int_0^{\cos(\frac{\pi}{N})} \frac{\frac{\pi}{2N}(1+t-t^{(N+2)})+\frac{3\pi}{2N}t^{(N+1)}}{(\cos(\frac{\pi}{N})-t)^2+(\frac{\pi}{N})^2}dt =$$

$$\int_0^{\cos(\frac{\pi}{N})} \frac{\frac{\pi}{2N}(1+t)+\frac{\pi}{2N}t^{(N+1)}(3-t)}{(\cos(\frac{\pi}{N})-t)^2+(\frac{\pi}{N})^2}dt = \widetilde{I_N}.$$

By changing the variable once again (to $\cos(\frac{\pi}{N})-t$) we obtain

$$\widetilde{I_N} = \int_0^{\cos(\frac{\pi}{N})} \frac{\frac{\pi}{2N}(1+\cos(\frac{\pi}{N})-\tau)+\frac{\pi}{2N}(\cos(\frac{\pi}{N})-\tau)^{(N+1)}(3-\cos(\frac{\pi}{N})-\tau)}{\tau^2+(\frac{\pi}{N})^2}d\tau.$$

Again up to the terms of greater order, we have to study

$$\int_0^{\cos(\frac{\pi}{N})} \frac{\frac{\pi}{2N}(2-\tau) + \frac{\pi}{2N}(\cos(\frac{\pi}{N})-\tau)^{(N+1)}(2-\tau)}{\tau^2 + (\frac{\pi}{N})^2} d\tau =$$

$$\int_0^{\cos(\frac{\pi}{N})} \frac{\frac{\pi}{2N}(2-\tau)}{\tau^2 + (\frac{\pi}{N})^2} d\tau + \int_0^{\cos(\frac{\pi}{N})} \frac{\frac{\pi}{2N}(\cos(\frac{\pi}{N})-\tau)^{(N+1)}(2-\tau)}{\tau^2 + (\frac{\pi}{N})^2} d\tau = J_N + G_N.$$

For any fixed $\epsilon > 0$,

$$J_N \geq \int_0^{\epsilon} \frac{\frac{\pi}{2N}(2-\tau)}{\tau^2 + (\frac{\pi}{N})^2} d\tau \geq \frac{2-\epsilon}{2} \int_0^{\frac{N\epsilon}{\pi}} \frac{1}{\tau^2+1} d\tau \to \frac{(2-\epsilon)\pi}{4}.$$

As ϵ can be any positive number, we see that $\liminf_{N\to\infty} J_N \geq \frac{\pi}{2}$.

If we recall the chain of the estimates, we see that $S_N(\frac{\pi}{N}) \geq 1 + \liminf_{N\to\infty} G_N$.

In the similar manner as with J_N, we see that

$$G_N \geq \int_0^{\frac{1}{N}} \frac{\frac{\pi}{2N}(\cos(\frac{\pi}{N})-\tau)^{(N+1)}(2-\tau)}{\tau^2 + (\frac{\pi}{N})^2} d\tau \geq$$

$$\int_0^{\frac{1}{N}} \frac{\frac{\pi}{2N}(1-\frac{\pi^2}{2N^2}-\frac{1}{N})^{(N+1)}(2-\frac{1}{N})}{\tau^2 + (\frac{\pi}{N})^2} d\tau = \int_0^{\frac{1}{N}} \frac{\frac{\pi}{N}e^{-1}}{\tau^2 + (\frac{\pi}{N})^2} d\tau + o(1) = e^{-1} \int_0^{\frac{1}{\pi}} \frac{1}{\tau^2+1} d\tau + o(1).$$

We conclude that $\liminf_{N\to\infty} G_N \geq e^{-1} \arctan(\frac{1}{\pi}) > 0$, which completes the proof (as $\sup(S_N) \geq S_N(\frac{\pi}{2N}) \geq 1 + e^{-1}\arctan(\frac{1}{\pi}) + o(1)$).