March 18, 2019 Topology Solutions 1

1 (a) True. $\bar{A} \subseteq A \cup B \subseteq \overline{A \cup B}$. Hence $\bar{A} \subseteq \overline{A \cup B}$.

Similarly $\bar{B} \subseteq \overline{A \cup B} \Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Next, $A \cup B \subseteq \bar{A} \cup \bar{B}$. Since $\bar{A} \cup \bar{B}$ is closed,

$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

(b) False. $A = (-1,0)$, $B = (0,1)$

LHS $= \phi$ RHS $= \{0\}$.

(c). $(C \times D)^c = ((X - C) \times Y) \cup (X \times (Y - D))$

C, D closed $\Rightarrow$ RHS open $\Rightarrow$ $C \times D$ closed.

(d). True. $A \times B \subseteq \bar{A} \times \bar{B}$. (c) $\Rightarrow$ RHS closed

and so $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$. For the other direction,

let $(a,b) \in \bar{A} \times \bar{B}$. Choose a basis element $U \times V$

~~$X \times Y$~~ for $X \times Y$ containing (a,b).

$a \in U$, $a \in \bar{A} \Rightarrow U \cap A \neq \phi$. Similarly $V \cap B \neq \phi$

which $\Rightarrow (U \times V) \cap (A \times B) \neq \phi$. Hence $(a,b) \in \overline{A \times B}$.

2 (a): The collection of all balls $\{B(x,\varepsilon) : x \in X, \varepsilon > 0\}$

forms a basis for a topology.

(b). No. $X = ((-\infty,\infty)$, usual metric$)$, $Y = ((0,1)$, usual metric$)$

(c) R^ω in box topology. In a metric space,

$x \in \bar{A}$ implies there is a sequence $(a_n) \subseteq A$ s.t. $a_n \to x$.

But if $A = \{(w_i) : w_i > 0\}$, then $\vec{0} = (0, 0, 0, \dots) \in \bar{A}$

but no sequence in A converges to $\vec{0}$.

3(a) Thm 26.3 in book

(b) $[0,1]$ with indiscrete topology.

$[0,1/2]$ compact but not closed.

(c) Thm 26.6 in book.

$f: X \to Y$. If C closed in X, then C compact in $X \Rightarrow f(C)$ compact in Y $\Rightarrow f(C)$ closed in Y. Hence (a)

f^{-1} is cont. and so f is a homeo.

(d) Let $f: [0,1)$ to S^1

$x \to e^{2\pi i x}$.

4(a). Yes. $d(x,y) = \sup_n \left\{ \frac{|x_n - y_n|}{n} \right\}$

A short explaination along the lines of the proof of Thm 20.5 suffices.

(b) No. There are many proofs. $\{0,1\}^N$ is a closed subset of $[0,1]^N$ and so one can show $\{0,1\}^N$ is not compact. $\{B(x, 1/4) : x \in \{0,1\}^N\}$ is an open covering of $\{0,1\}^N$. These sets are disjoint and so there is no finite subcovering.

(c) Yes. This was proved in class.

A short explanation along those lines suffices.

d. In the product topology, $\overline{A} = [0,1]^{\mathbb{N}}$. One needs to show each open set intersects A. It suffices to show each basis element for the product topology intersects A. This is immediate. In the uniform topology,

$$\overline{A} = \{ x \in [0,1]^{\mathbb{N}} : \lim_{i\to\infty} x_i = 0 \}.$$

If $\lim_{i\to\infty} x_i \neq 0$, then there exists $\varepsilon > 0$ so that $B(x,\varepsilon) \cap A = \phi \Rightarrow x \notin \overline{A}$.

If $\lim_{i\to\infty} x_i = 0$, then $\forall \varepsilon > 0$, $B(x,\varepsilon) \cap A \neq \phi$
$\Rightarrow x \in \overline{A}$.

5(a). We have a natural map π from $\mathbb{R}^2$ to $\mathbb{R}^2/\sim$ sending x to $[x]$. We then put the quotient topology on $\mathbb{R}^2/\sim$; $A \subseteq \mathbb{R}^2/\sim$ is open iff $\pi^{-1}(A)$ is open in $\mathbb{R}^2$.

(b) Letting $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = xy$, Y is just the quotient space obtained using f; ie $(x,y) \sim (x',y')$ if $f(x,y) = f(x',y')$. Since $f^{-1}\{a\}$ is closed in $\mathbb{R}^2$ $\forall a$, T1 follows.

(c)

$$\begin{tikzcd} \mathbb{R}^2 \arrow[d, "\pi"'] \arrow[dr, "f(x,y)=xy"] & \\ \mathbb{R}^2/\sim \arrow[r, dashed] & \mathbb{R} \end{tikzcd}$$

From Cor 22.3, $\exists g : \mathbb{R}^2/\sim \to \mathbb{R}$ which is a homeomorphism (and diagram commutes) if f is a quotient map. However, easy to see f is an open map and so a quotient map.

6(a) $f \sim g$ if $\exists\, F : S^1 \times [0,1] \to X$ cont. so that $F(s,0) = f(s)$, $F(s,1) = g(s)$ $\forall s \in S^1$.

(b) $f \sim_p g$ if ~~for~~ $\exists\, F : [0,1] \times [0,1] \to X$ cont. so that $F(s,0) = f(s)$, $F(s,1) = g(s)$ $\forall s \in [0,1]$, $F(0,t) = F(1,t) = a$ $\forall t \in [0,1]$

(c) At time $1/2$, you will have transversed γ_1 and γ_2 in first case and only γ_1 in 2nd case. For the fund. group, these are homotopic and so identified.

(d) $X = \mathbb{R}^2 \setminus \{0\}$. $a = (1,0)$. $\gamma_1(t) = e^{2\pi i t}$, $\gamma_2(t) = e^{-2\pi i t}$.

Not path homo. since different elements in fund group. Clearly homotopic even where we can keep the path at 0 to be a.

7(a). $r : X \to A$ so that $r(a) = a$ $\forall a \in A$

(b) $A \overset{}{\underset{inc.}{\hookrightarrow}} X \underset{r}{\to} A$. $r \circ inc$ = identity

$\Rightarrow$ $r^* \circ (inc)^*$ = identity on groups $\Rightarrow$ $(inc)^*$ injective

(c). $X = S^1$, $A = \{(1,0)\}$.

(d) $\exists F : X \times I \to X$ cont. so that $F(x,0) = x$ $\forall x$ $F(x,1) \in A$ $\forall x$, $F(a,t) = a$ $\forall a$ $\forall t$.

The exercise we did provides lots of examples.

8(a) See book for defn. The example is not a cov. map since $(1,0)$ has no neighborhood evenly covered by the map; see Ex 2.

(b) See pg 342

(c) Let $f : S^1 \to S^1$ be the identity. This cannot be lifted since any map from S^1 to $\mathbb{R}$ collapses two points. If Y is $[0,1]$, then f can be lifted by Lemma 54.1.

(d) Let $C = \{n + 1/n : n \geq 2\}$. C closed. But $\pi(C)$ has $(1,0)$ as a limit point, but $(1,0) \notin \pi(C)$.