Solution, Exam in MMA 100 Topology, 2014-06-12.

1. Let X be a topological space. Prove or disprove the following statements for subsets of X:

   (a) $\overline{A} \cap \overline{B} \subset \overline{A \cap B}$. ($\overline{A}$ stands for the closure of A).

   (b) $A^o \cap B^o = (A \cap B)^o$. ($A^o$ stands for the subset of interior points of A).

   Solution. (a). False. Let $A = (0, \infty)$ and $B = (-\infty, 0)$.

   (b) True. Proof: If $x \in LHS = A^o \cap B^o$, then there exist open U and V such that $x \in U \subset A, x \in V \subset B$. Thus $x \in U \cap V \subset A \cap B$, and $U \cap V$ is open, namely $x \in RHS = (A \cap B)^o$ by def.. Now if $x \in RHS$ then there is U open $x \in U \subset A \cap B$, so $x \in U \subset A$ and $x \in U \subset B$, and consequently $x \in A^o \cap B^o = LHS$.

2. We define three topological spaces as follows: $A = \mathbb{R}^2/\sim$ is the identification space by identifying the subset $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$ as a single point and each other point of $\mathbb{R}^2$ as itself, $B \subset \mathbb{R}^3$ is the union of spheres of radius $\frac{1}{n}$ tangential to the xy-plane at the origin (a spherical earring), and $C \subset \mathbb{R}^3$ is the union of spheres of radius n tangential to the xy-plane at the origin. Find homeomeorphic and non-homeomorphic pairs among the spaces A, B and C. Provide brief geometric arguments.

   Solution. A and C are homeomorphic, A and B are not homeomorphic (and so is also the pair (B, C)).

   No homeomorphic pairs: B is a compact set but A and C are non-compact.

   Homeomorphic pairs: Let $\mathbb{R}^2 \to A$ be the identification map. The plane $\mathbb{R}^2$ is a union of unit squares, enumerated by U_n. First we identify each square with a sphere S_n with its boundary being glued together as a marked point p_n. Denote q the distinguised point in A obtained by identifying the integral lines. We define a map f from A to C by sending p_n to the origin and sending the spheres S_n to the sphere of radius n. f is bijective and a homeomorphic mapping.

3. Let (X, d) be a metric space. Prove that if $C \subset X$ is closed and $p \notin C$ then there exists a continuous function $f : X \to \mathbb{R}$ such that $f = 0$ on C, i.e. $f(c) = 0, \forall c \in C$, and $f(p) = 1$.

   Solution.

$$f(x) = \frac{d(x, C)}{d(x, p) + d(x, C)}$$

4. A continuous map $f : X \to Y$ is called *proper* is $f^{-1}(C)$ is compact for any compact set $C \subset Y$.

   [a] Prove that any proper map $f : \mathbb{R}^n \to \mathbb{R}^m$ is unbounded.

   [b] Suppose X is compact and Y is Haussdorff. Prove that any continuous map $f : X \to Y$ is proper.

   Solution. (a) The closure of the image $f(\mathbb{R}^n)$ is not compact, otherwise its preimage should be compact since f is proper, but the preimage is $\mathbb{R}^n$ which is non-compact. Thus the closure is unbounded, so is also $f(\mathbb{R}^n)$ itself.

   (b) Suppse $C \subset Y$ is compact. Then C is closed since Y is Haussdorff. Consequently $f^{-1}(C)$ is closed in X. But X is compact and a closed subset of X is compact, namely $f^{-1}(C)$ is compact.

5. Let $Func(\mathbb{R}^n)$ be the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$. Let $M_{C,U} = \{f \in Func(\mathbb{R}^n); f$ maps C into $U\}$, where $C \subset \mathbb{R}^n$ is compact and $U \subset \mathbb{R}$ is open. We define a topology on $Func(\mathbb{R}^n)$ by requiring that the subsets $\{M_{C,U}\}_{C,U}$ form a basis. Prove that (**a**) $Func(\mathbb{R}^n)$ is Haussdorff, and (**b**) If $\lim_{n\to\infty} f_n = f$ in $Func(\mathbb{R}^n)$ then $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}^n$.

   Solution. (a) If $f \neq g$ in $Func(\mathbb{R}^n)$ then $f(x_0) \neq g(x_0)$ for some x_0. Denote $2\varepsilon = |f(x_0) - g(x_0)|$. Consider the neighborhoods $U = \{y \in ;|y - f(x_0)| < \varepsilon\}$, $V = \{y \in ;|y - g(x_0)| < \varepsilon\}$ and the compact set $C = \{x_0\}$. Then the neighborhood $M_{C,U}$ contains f and $M_{C,V}$ contains g, and they are disjoint.

   (b) Suppose $\lim_{n\to\infty} f_n = f$ in $Func(\mathbb{R}^n)$. For any x_0 and any ε we consider the set $C = \{x_0\}$ and $U = \{y \in ;|y - f(x_0)| < \varepsilon\}$. So there exists n_0 such that $f_n \in M_{C,U}$ for $n \geq n_0$, namely $|f_n(x_0) - f(x_0)| < \varepsilon\}$, for for $n \geq n_0$.

6. Prove that if X and Y have the same homotopy type then $\pi_1(X) = \pi_1(Y)$, i.e. isomorphic.

   Solution. See the textbook

7. Let $\mathbb{P}^3$ be the projective space of lines $[x] = \mathbb{R}x$ in $\mathbb{R}^4$. We write any vector x in $\mathbb{R}^4$ as $x = (u, v)$, $u, v \in \mathbb{R}^2$ and consider the subset $X = \{[x] \in \mathbb{P}^3; x = (u, u), 0 \neq u \in \mathbb{R}^2\}$. Find the homotopy group of $\mathbb{P}^3 \setminus X$. (Hint: Consider the map $[u, v] \to [u - v] \in \mathbb{P}^1 = S^1$ and the homotopy $h(t, [u, v]) = [u - tv, (1 - t)v]$.)

   Solution. Denote $A := \mathbb{P}^3 \setminus X$ and consider the maps $f : A \to \mathbb{P}^1, [u, v] \mapsto [u - v] \in \mathbb{P}^1 = S^1$ and $g : \mathbb{P}^1 \to A, [u] \mapsto [u, 0]$. Then $f \circ g$ is the identity on $\mathbb{P}^1$ and $h(t, [u, v]) = [u - tv, (1 - t)v]$ defines then a homotopy between $g \circ f$ to the identity $I : A \to A$. Thus A and $\mathbb{P}^1$ are of the same homotopy type.

8. Consider the map $f : \mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, $f(u, v, w) = ((u \times v) \times w, (u \cdot v)w, (u \times v) \cdot w)$. Prove that f induces a well-defined map from $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^6$. Describe the induced map on the homotopy groups. (Reminder: $u \times v$ is the cross-product/vector-product in $\mathbb{R}^3$.)

   Solution. Let u, v, w be any three non-zero vectors. If they are linearly independent then the last term is nonzero since it is the volyme of the parallellopiped generated by them. Now suppose they are linearly dependent, then they are in a common plane. If $u \perp v$ they the second term is zero but the first term is non-zero since $u \times v$ is a non-zero vector orthogonal to w. If u is not perpendicular to v then the second term is non-zero. This proves that f maps any non-zero vectors u, v, w to a non-zero vector.

   Next the map f is trilinear and it thus induced a map on the projective spaces

   The nonzero element -1 in $\pi_1(\mathbb{R}^3)$ is representated by a simple curve starting from $e_1 \to -e_1$ and we may choose it to be the half circel in the upper half plane of xy-plane; its square is the trivial element 1 represented by the circle from e_1 to e_1 with double speed. The trilinear map f then maps the homotopy class $(-1, -1, -1)$ to -1 and $(-1, -1, 1)$ to 1. Using the symmetry we conclude that f_* is the map $Z_2 \times Z_2 \times Z_2 \to Z_2, (a, b, c) \to abc$.