Exercises (1 and 4 are assignments, the deadline for submitting the solution is Friday Nov 16.

1. $*$ (Finite dimensional Banach spaces.) (1) Consider the space $l^p(\mathbb{R}^n) = \mathbb{R}^n$, $1 \leq p \leq \infty$, equipped with the norm

   $$\|x\|_p^p = \sum_{i=1}^{n} |x_i|^p,$$

   $\|x\| = \|x\|_2$ being the standard Euclidean norm. Find estimates for the constants $c = c(n)$ and $C = C(n)$ such that

   $$c\|x\| \leq \|x\|_p \leq C\|x\|, x \in \mathbb{R}^n.$$

   Is $c(n) \to \infty$, $C(n) \to \infty$, as $n \to \infty$? (It's difficult to find the exact value of $c(n)$ and $C(n)$, but you can still answer this question.)

   (2) $*$ Consider now the space $l^p = l^p(\mathbb{N})$, $1 \leq p \leq \infty$,

   $$\|x\|_p^p = \sum_{i=0}^{\infty} |x_i|^p.$$

   For which pairs of (p, q), is the identity map $I : x \to x$ a bounded operator $I : l^p \to l^q$?

2. (Non-compactness of the unit ball in an infinite-dimensional Banach space. See the textbook for the slightly abstract proof.) (1) Let $0 < \epsilon < 1$ be a fixed number. Let V be a Banach space and $W \subsetneq V$ a finite-dimensional vector subspace in V. Prove that there exists a unit vector $v \notin W$, $\|v\| = 1$, and $1 \geq \text{dist}(v, W) \geq \frac{1}{1+\epsilon}$.

   (2) Construct a sequences of infinite sequence of unit vectors $\{v_n\}$ such that $1 \geq \text{dist}(v_{n+1}, W_n) \geq \frac{1}{1+\epsilon}$, , $\|v_{n+1}\| = 1$. In particular v_n has no convergent subsequences.

3. (S_p-norm of matrices). Consider $\mathbb{R}^n = (\mathbb{R}^n, \|\cdot\|)$ the Euclidenn spaces and $M_{mn}(\mathbb{R})$ of real matrices T, $m \leq n$ and linear maps from $\mathbb{R}^n \to \mathbb{R}^m$. Recall Linear Algebra that T^tT is a $m \times m$ symmetric matrix (and TT^t is a $n \times n$-symmetric matrix) and $T^tT \geq 0$ (resp. $TT^t \geq 0$). As symmetric matrix TT^t is diagonalizable and has nonnegative eigenvalues, multiplicity counted, $\lambda = (\lambda_1, \cdots, \lambda_m) : \lambda_1 \geq \lambda_2 \geq \cdots \lambda_m \geq 0$. We define the S_p-norm (similar to l^p-norm) as

   $$\|T\|_p = \|\lambda\|_{l^p}$$

   Prove that $\|T\|_\infty$ is the matrix-norm of $T : \mathbb{R}^n \to \mathbb{R}^m$ and $\|T\|_2$ is a Hilbert-space norm, i.e. obtained from an inner product. (All $\|T\|_p$ are also norms, $1 \leq p \leq \infty$, but requires some effort to prove.)

4. $*$ Let $c > 0$ and T be the following matrix

$$T = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

(1) Find the norm of T as a map $T : (\mathbb{R}^2, \|\cdot\|_2) \to (\mathbb{R}^2, \|\cdot\|_p)$, $1 \le p \le \infty$. (2) Find the norm $\|T\|$ of $T : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_p)$.

5. (Examples of unbounded linear functionals defined on dense subspace of Banach spaces). Consider the subspace D of $l^\infty(\mathbb{N})$ consisting of sequences $x = (x_n)$ with finite supports, i.e., $x_n = 0$ for n sufficiently large (depending on x). Let $\lambda : D \to \mathbb{R}$, $\lambda(x) = \sum_n x_n$. Prove that (1) D is dense in $l^\infty(\mathbb{N})$ (in particular D is not a Banach space sinc e $D \neq l^\infty(\mathbb{N})$). (2) λ is unbounded on D and can not be extended to $l^\infty(\mathbb{N})$.

6. (Examples of unbounded linear functionals). Most linear functions on a Banach space X (not on a dense subspace) are continuous, i.e bounded. Indeed, it is rather difficult to construct unbounded linear functionals which are defined on the whole space X.

   (1) Use Zorn's lemma (that any partially ordered set has a maximal element) to proof the following claim: On any vector space X there is a linear basis $B = \{b_\alpha\}$, in the sense that any $x \in X$ is a linear combination $x = \sum x_\alpha b_\alpha$ of B (all coefficient being zero except finitely many). (This is also called Hamel's basis.)

   (2) Let $X = l^\infty(\mathbb{N})$. Let $\{e_n\}$ be the "standard basis" vector. Clearly $\{e_n\}$ is not a linear basis of $X = l^\infty(\mathbb{N})$, so there is a Hamel's basis $\{b_\alpha\}$ containing the "standard basis" vectors $\{e_n\}$. Let b_{α_0} be any fixed basis vector not in $\{e_n\}$. Any vector $x \in X$ can be written as $x = \sum_\alpha x_\alpha b_\alpha$, and we define $\lambda : x \mapsto x_{\alpha_0}$. Prove that λ is unbounded.