Exercises (Nos. 1 and 4 are assignments, to be submitted by Friday Nov 23.)

1. (Hilbert space vs Banach spaces) (1) Let H be a Hilbert space. Prove that $\|\cdot\|$ is additive for two vectors $v \neq 0$ and $w \neq 0$, i.e. $\|v+w\| = \|v\| + \|w\|$ iff v and w are in a ray, i.e. $v = cw$ for some $c > 0$.

   (2) Let $(\mathbb{R}^n, \|\cdot\|_\infty)$ be equipped with the l^∞-norm and $\mathbb{E}^n = \mathbb{R}^n$ be the Euclidean space with the l^2-norm. Define $T : \mathbb{E}^n \to (\mathbb{R}^n, \|\cdot\|_\infty)$, by $T0 = 0$ and $T(x) = \frac{\|x\|_2}{\|x\|_\infty} x$, $x \neq 0$. Observe that $\|Tx\|_\infty = \|x\|_2$. Is T continuous? Can you find a **linear** isometry S, i.e. $\|Sx - Sy\|_\infty = \|x - y\|_2$ from $\mathbb{E}^n$ to $(\mathbb{R}^n, \|\cdot\|_\infty)$? Provide as simple arguments as possible.

   (3) Observe that $l^2 \subset\subset l^\infty$. Is the identify map $Id : l^2 \subset\subset l^\infty$ bounded? Consider the same questions as in (2), i.e, is the operator $T : l^2 \to l^\infty$, $T(x) = \frac{\|x\|_2}{\|x\|_\infty} x$, continuous? Does there exists a linear isometry from l^2 onto l^∞? (To understand whether l^2 and l^∞ are isomorphic as Banach spaces is not a very easy question.)

2. Recall that the set of rational numbers in $\mathbb{R}$ is a countable dense subset. Thus $\mathbb{R}^n$ has also a countable sense subset since product of countable sets is countable. Prove that l^∞ has no countable dense subset.

3. (Convex sets, locally convex vector spaces; see Chapter 8) Unit ball in Banach spaces are special kind of convex sets. We describe how Banach norms on $\mathbb{R}^n$ can be described and constructed using convex sets.

   We equipp $\mathbb{R}^n$ with its underlying Euclidean norm and topology (as a reference, any other equivalent norm will work). Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

   (1) Check that the closed unit ball $B = B_1(0)$ satisfies the following property: (a) B is convex and bounded, (b) 0 is an inner point of B, (c) B is balanced, i.e. if $x \in B$ then $\lambda x \in B$ for all $|\lambda| \leq 1$, and (d)

   $$\|x\| = \inf\{t > 0; \frac{1}{t} x \in B\} = \inf\{t > 0; x \in tB\}.$$

   (2) Suppose now B is a set in $\mathbb{R}^n$ satisfying (a), (b), and (c). Prove that

   $$\|x\|_B := \inf\{t > 0; x \in tB\}$$

   defines a norm.

   (3) (Minkowski sum of convex sets.) Let A and B be two convex sets in $\mathbb{R}^n$. The Minkowski sum $A + B$ is defined as the set

   $$A + B = \{a + b; a \in A, b \in B\}.$$

   It is clear that $A + B$ is also convex. Suppose A and B satisfies the conditions (a)-(b)-(c) above. Prove that $A + B$ satisfies also the conditions. In particular it also defines a norm on $\mathbb{R}^n$. (Is that the norm $\|x\|_A + \|x\|_B$?)

4. (This is to motivate the Banach-Steinhauss theorem.) Let g_n be the following functions on $[0,1]$, $n = 1, 2, \cdots$,

$$g_n(x) = \sum_{k=1}^{n} \frac{n}{k} \chi_{[\frac{2k}{2n+1}, \frac{2k+1}{2n+1}]},$$

where χ_A stands for the characteristic function of a set A. Define

$$T_n f = \int_0^1 f(x) g_n(x) dx$$

(1) Prove that T_n is a bounded linear functional on $C[0,1]$.

(2) Find the norm of T_n and prove that $\{\|T_n\|\}$ is unbounded.

(3) Find a function f so that $\{T_n f\}$ is divergent and one $f \neq 0$ such that $\{T_n f\}$ is convergent.

5. $*$ (1) We observe that the "unit disc" in $(\mathbb{R}^2, \|\cdot\|_1)$ (and in $(\mathbb{R}^2, \|\cdot\|_\infty)$) is a square, and the norm in $\|\cdot\|_1)$ is additive on each of the four quadrant. Prove that this is true for general norms obtained for convex polyhedra, namely if the unit disc in $(\mathbb{R}^2, \|\cdot\|)$ is a convex polyhedron then $\mathbb{R}^2$ is a union of cones where the norm on each cone is additive:

$$\|x + y\|_\infty = \|x\|_\infty + \|y\|_\infty$$

for x and y in a common cone.

(2) Suppose $(\mathbb{R}^2, \|\cdot\|)$ is equipped with a Banach norm such that the unit ball is a regular hexagon around the origin. The unit disc of its dual space $(\mathbb{R}^2, \|\cdot\|)$ (equipped with the defining norm) is also a convex polygon. Describe and draw the convex polygon. (Use the fact that a linear function $f(x) = a_1 y_1 + a_2 y_2$ is constant on lines orthogonal to the vector (y_1, y_2), and use elementary geometric arguments.) Convince your self that the double dual $(V^*)^*$ of $V = (\mathbb{R}^2, \|\cdot\|)$ equals (i.e. isometric) to V.