Exercises, Week 3. (Ex. 3 and 5 are assignments, to be submitted by Mån. Dec 3.)

(1) Suppose $Y \subset X$ is a closed linear subspace of a Banach space X. Prove that $Y + \text{Span}\{x_1, \cdots, x_n\}$ is closed for any linearly independent vectors $x_j \notin Y$.

(2) Suppose X is a Banach space and $\phi : X \to \mathbb{R}^n$ is a linear map. (a) Prove that ϕ is bounded if and only if $\text{Ker}\,\phi$ is closed

(b) Suppose ϕ is bounded. Prove that X has a direct sum decompostion: There is a closed subspace Z such that $Z \cap \text{Ker}\,\phi = 0$ and $X = Z \dot{+} \text{Ker}\,\phi$, i.e, each x has a unique decomposition $x = z + k$ for some $z \in Z, k \in \text{Ker}\,\phi$.

(3) (This is somewhat dual to the Exercise above but somewhat more difficult.). Let $T : X \to Y$ be a bounded linear operator and onto.

(a) Prove that the induced map $\tilde{T} : X/\text{Ker}\,T \to Y$ is a bounded and boundedly invertible operator.

(b) Suppose $\text{Ker}\,T$ is finite dimensional. Prove that X has also a direct sum decomposition $X = \text{Ker}\,T \dot{+} X_0$, with X_0 is isomorphic to Y as Banach spaces (i.e., there exists a bounded and boundedly invertible $T : X_0 \to Y$).

(c) Suppose X is Hilbert spaces and $T : X \to X$ is a bounded linear operator such that $\text{Ker}\,T$ is finite-dimensional and $\text{Im}\,T$ is co-finite dimensional, i.e $X/\text{Im}\,T$ is finite dimensional. Prove that ImT is closed. Prove that $ind(T) = \dim \text{Ker}\,T - \dim X/\text{Im}\,T$ is invariantly defined up to finite rank pertubation, i.e $ind(T) = ind(T + S)$ if S is any finite rank operator. (Start with a rank one operator $S : X \to X, Sx = \langle x, x_0 \rangle y_0$ for some $x_0, y_0 \in X$.) (Such T is called Fredholm.)

(4) Let $c_0 \subset l^\infty$ be the subspace of sequences $x = (x_n), x_n \to 0, n \to \infty$. Find its dual space.

(5) (Positive linear functional on Banach space and Hahn-Banach theorem). Consider the spaces $X = C[a, b]$ and l^∞. A linear functional $\phi : X \to \mathbb{R}$ is called positive is $\phi(f) \geq 0$ for $f = f(t) \geq 0, t \in [0, 1]$.

(0) (This is not assignment) Which of the following functionals are positive: (a) $\phi(f) = f(0)$, (b) $\phi(f) = \int_a^b f(t)dt$, (c) $\phi(f) = f(1) - f(0)$

(1) Prove that if ϕ is positive then ϕ is bounded and $\|\phi\| = \phi(1)$, where 1 is the constant function. Prove that any bounded linear functional ψ on l^2 can be written as $\psi = \phi_1 - \phi_2$ where ϕ_1 and ϕ_2 are positive.

(2) Consider the subspace $l_{lim} = \{x = (x_n), \exists \lim_{n\to\infty} x_n\}$ of l^∞. It is clear that $\phi(x) = \lim x_n$ is a bounded linear functional on l_{lim}. Extend ϕ to two bounded linear functionals ϕ_1 and ϕ_2 on l^∞ such that $\phi_1(x) = 1$ and $\phi_2(x) = -1$ for the sequence $x = (-1)^n$.