Exercises, Week 4. (Ex. 1 and 4 are assignments, to be submitted by Wednes. Dec 12.)

(1) (REVISED. THE EARLIER FORMULATION IS NOT CORRECT). (a) Prove that there exists an mean ψ on $l^\infty(\mathbb{Z})$ (i.e. positive functional such that $\psi(1) = 1$) which is invariant under the reflection $R : (x_n) \to (x_{-n})$ and shift $T : (x_n) \to (x_{n+1})$. (Hint: Let ϕ be a Banach limit on $l^\infty(\mathbb{N})$. Consider the following operator $A : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{N})$,

$$(Ax)_n = \frac{1}{2n+1} \sum_{|j| \leq n} x_j, \quad n = 0, 1, 2, \ldots.$$

(b) Consider the subspace $l^\infty_{lim}(\mathbb{Z})$ of sequences $x = (x_n)_{-\infty}^{\infty}$ such that both $\lim_{n\to\pm\infty} x_n$ exist. Define the linear functional λ on $l^\infty_{lim}(\mathbb{Z})$ by

$$\nu(x) = \frac{1}{2} \lim_{n\to\infty} (x_n + x_{-n}).$$

Extend λ to a linear functional Λ on $l^\infty(\mathbb{Z})$. Is Λ reflection invariant? translation invariant? If not, can you redefine it by pre-composing a bounded map on $l^\infty(\mathbb{Z})$ to make it both translation and reflection invariant? (See the proof of Banach-limit construction on $l^\infty(\mathbb{N})$.)

(2) Make a statement on the relation between w^*-closed sets and norm-closed sets in X^*, i.e. decide whether (1) w^*-closedness implies norm-closedness or vice versa. Prove your statement and provide a counter example for the converse of your statement.

(3) (a) Let $\{x_j\}_1^n$ be linearly independent vectors in a normed vector space X and $c_1, \cdots, c_n \in \mathbb{C}$. Prove that there exists $f \in X^*$, such that $f(x_j) = 0$. (b) Let $\{x_j\}_1^n$ be as above. Prove that the subspace $Y = \{\phi \in X^*, \phi(x_j) = 0, j = 1, \cdots, n\}$ is closed subspace of X^* (in both $w*$ and norm topology) and prove that there is a decomposition $X^* = Y \dot{+} \mathbb{C}^n$.

(4) Let $1 \leq p \leq \infty$. Consider the following subspace of l^p: $Y = \{x : \sum_n x_n = 0\}$. (Here $\sum_n x_n$ is defined in its orginal meaning, namely $\lim_{N\to\infty} \sum_0^N x_n$ exists, so that $\sum_n |x_n|$ might diverge.)
(a) Prove that Y is a dense subspace of l^p, for $1 < p < \infty$.
(b) Let $p = 1$. Prove Y is a closed subspace.
(c) Let $p = \infty$. Is Y dense? Is Y closed?