Exercises, Week 5. (Ex. 1 and 3 are assignments, to be submitted by Wedn. Dec 19.)

(1) (cf Riesz-Thorin) Consider the spaces $L^p[0,1]$ with the Lebesgue measure. Let $1 \leq q_0 < q_1$, $t \in (0,1)$, and $q = q(q_0, q_1, t)$ be determined by $\frac{1}{q} = \frac{(1-t)}{q_0} + \frac{t}{q_1}$. The Riesz-Thorin theorem states that $L^{q_0} \cap L^{q_1}$ is a subspace of L^q.

(a) Is $L^{q_0} \cap L^{q_1}$ a closed subspace of L^q? The Riesz-Thorin inequality states that $\|f\|_q \leq \|f\|_{q_0}^{1-t}\|f\|_{q_1}^{t}$. Is that possible to get an estimate $\|f\|_{q_0}^{1-t}\|f\|_{q_1}^{t} \leq C\|f\|_q$?

(b) For finite dimensional l^p-spaces $(\mathbb{R}^n, l^p)$, find an an estimate $\|f\|_{q_0}^{1-t}\|f\|_{q_1}^{t} \leq C\|f\|_q$ with optimal C.

(2) (Ameanable groups) (a) Prove that the Heisenberg group

$$H(\mathbb{Z}) = \{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, a, b, c \in \mathbb{Z}\}$$

is ameanable, the group operation being the matrix product.

(3) Consider the space $X = l^\infty([1,\infty))$ and its subspace $Y = C_{lim}([1,\infty))$ of continous functions $f(x)$ such that $\lim_{x\to\infty} f(x)$ exists. (OBS! $C_{lim}([1,\infty))$ replacing $l^\infty([1,\infty))$ which was not properly defined, or one has to redefine $\lim_{x\to\infty} f(x)$ using ess-sup and ess-inf on $[x,\infty)$ and take their limit.) Define $lim_{x\to\infty}$ on Y and state a Banach limit theorem for X by using Hahn-Banach theorem and the "Cesaro" average $T : f \to \frac{1}{x}\int_1^x f(t)dt$.

(4) (Topologies on space of operators $B(H)$.) On the space $B(H)$ of bounded operators on a Hilbert space H there are three natural topologies defined by the semi-norms, $\|A - B\|$ (operator norm topology), $\|Ax - Bx\|$, $x \in H$ (strong topology), and $|\langle Ax - Bx, y\rangle|$, $x, y \in H$ (weak topology). Provide examples to show that they are not equivalent.

(5) Work out details that the tri-nary series $\sum_{n=1}^{\infty} \frac{x_n}{3^n}$ defines a continuous map from the totally disconnected set $\prod_1^\infty\{0,2\}$ onto the Cantor set.

(6) Ex. 2.50 and Ex. 8.32 (on equidistributions) in the book of Einsiedler and Ward.