MMA210, Advanced Differential Analysis
Assignment 4 , turn in: Course-week 7

1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. Consider the set
$$S_C = \{x \in \mathbb{R}^d : f(x) = C\}.$$
The set of C, for which S_C is nonempty, is an interval.
   (a) Prove that the set Σ of C, for which $\nabla f = 0$ for all $x \in S_C$, is a closed set of measure zero.
   (b) Prove that for $C \notin \Sigma$, S_C is an $n-1$ - dimensional orientable manifold. Find some conditions on f for S_C to be compact.
   (c) Prove that the volume (area) form on S_C is given by
$$\omega = \|\nabla f\|^{-1}(\partial_1 f dx^2 \wedge \cdots \wedge dx^{d-1} - \cdots + (-1)^d \partial_d f dx^1 \wedge \cdots \wedge dx^{d-1}).$$
   (d) Derive a formula that describes the relation between the volume of the unit ball in $\mathbb{R}^d$ and the unit sphere (use the question above and Stokes' theorem)
2. Show that the Möbius band in $\mathbb{R}^3$ cannot be represented as $S \cap U$ where S is the level-set of some smooth function and U is an open set in $\mathbb{R}^3$