MMA210, Higher Differential Calculus,
Assignment 5, to be returned Friday, the last week.

Let $\mathbb{T}$ be the unit circle in the plane (identified with the complex plane), with local co-ordinates defined as (U, θ), $\theta : U(\subset \mathbb{R}) \to \mathbb{T}$, $\theta(\phi) = \exp(i\phi)$, where U is any interval with length less than 2π.

(1) Prove that $\eta = d\phi$ is well defined on $\mathbb{T}$ (Obs! θ is not defined on the whole of $\mathbb{T}$)
let k be an integer and $g : \mathbb{T} \to \mathbb{T}$ is the mapping $z \mapsto z^k$. Prove that $\int_{\mathbb{T}} g^*\eta = 2\pi k$.
If $f : \mathbb{T} \to \mathbb{T}$ is a smooth mapping and $F : \mathbb{R} \to \mathbb{R}$ be its lift to $\mathbb{R}$, as in the problem of weeks 2-3, $F(s + 2\pi) = F(s) + 2\pi k$. Prove that f and g are homopically equivalent. Prove that $\int_T f^*\eta = 2\pi k$.

(2) Consider the torus T^2 with local co-ordinates $(x^1, x^2) \mapsto (e^{ix^1}, e^{ix^2})$. Prove that $\dim H^1(T^2) \geq 2$ (in fact, $\dim H^1(T^2) = \mathbb{R}^2$, prove this, if you can!)