18/1

ODE, dynamical systems
MMA421 / TMA0M

Info 1) Teachers: Bernt Wennberg www.math.chalmers.se/~wennb
Dawan Mustafa
Alexei Heintz (week 8)

2) Please write name + e-mail on list

3) Register: If you haven't already:
GU-students: studera.nu within 2 weeks
(ask friend for help, or Jeanette Montell)

Chalmers student: studieportalen
https://student.gate.chalmers.se
within 24/1 17:00

4) Course organisation:
4 h lecture + 2 h problems / labs

lab see web page ...
note that it will be graded together with exam. Advisable to let me check a preliminary version before.

Demonstrations You should sign up for problems ....

5. About the web ....
www.math.chalmers.se / ~wennberg

6. $\boxed{\text{7.5 hp} \approx \text{5 weeks} \approx \text{200 hours!}}$

Introduction

"A dynamical system is a semigroup G acting on a space M"

There is a map $T : G \times M \to M$

$$(g, x) \longmapsto T_g x$$

such that

$$T_g \circ T_h = T_{g \circ h}$$

[who understands this? If not, you will!]

Examples

Discrete: $G = (\mathbb{N}, +)$ $\{0, 1, 2, 3, \dots\}$

$(\mathbb{Z}, +)$ $\{\dots, -3, -2, -1, 0, 1, 2, \dots\}$

$M = \mathbb{R}$

Iterated maps: $x_0 \curvearrowright x_1 \curvearrowright x_2 \curvearrowright \dots$

$$x_{k+1} = f(x_k), \quad k \geq 0$$

$$f : \mathbb{R} \to \mathbb{R}$$

$$T_0 = Id \quad (T_0 x = x)$$

$$T_k = f^k = \underbrace{f \circ f \circ \dots \circ f}_{k \text{ times}} = f(f(\dots f(\cdot) \dots))$$

$$T_k \circ T_n(x) = f^k(f^n(x)) =$$

$$\underbrace{f(f(\dots f}_{k}(f^n(x) \dots) = \underbrace{f(f(\dots f(x) \dots}_{k+n}) = f^{k+n}(x)$$

$$= T_{k+n}(x)$$

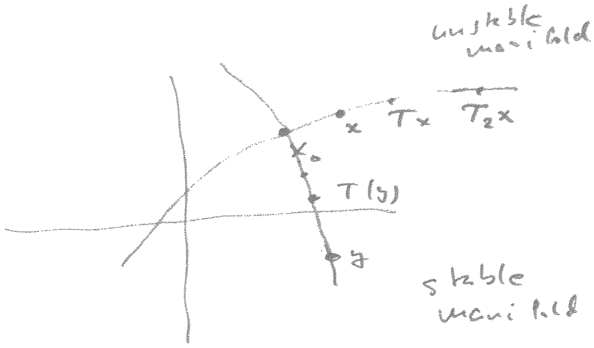
What are interesting questions to ask?

Ex 1) what happens to $T_k(x)$ when $k \to \infty$?

2) what sets $A \subset M$ satisfy

$$x \in A \Rightarrow T_k(x) \in A \quad \text{for all } k \in G ?$$

3) If $T_1(x_0) = x_0$ what happens to x close to x_0?



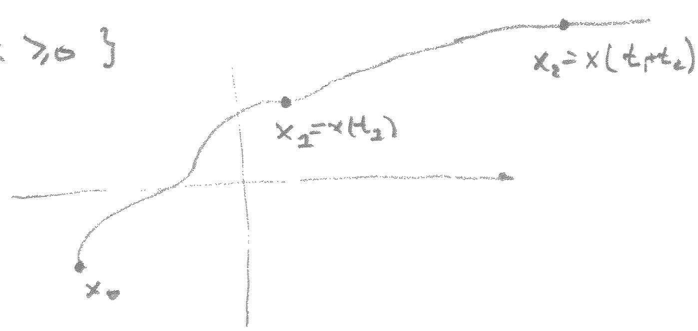
4) what happens if we perturb T_k?

Continuous systems ODE

$$\begin{cases} \dot{x} = f(x) & x \in M \quad (= \mathbb{R}^n, \text{ often}) \\ x(0) = x_0 \end{cases}$$

Let the solution be $x(t) = \Phi^t(x_0)$

Here $G = \mathbb{R}^+ = \{ x \in \mathbb{R}, \ x \geq 0 \}$



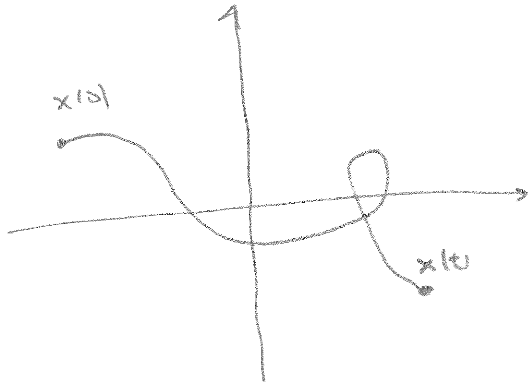
$$x(t_1+t_2) = \Phi^{t_1+t_2}(x_0)$$

$$= \Phi^{t_2}(x_1) = \Phi^{t_2}(\Phi^{t_1}(x_0))$$

$$= \Phi^{t_2} \circ \Phi^{t_1}(x_0)$$

Same kind of questions are interesting.

Ordinary Differential equations



Notation $x(t): \mathbb{R} \to \mathbb{R}^m$

$$\frac{dx}{dt} = \dot{x} = x^{(1)}$$

$$x^{(k)} = \frac{d^k x}{dt^k}$$

An ODE is an equation

$$\boxed{F(t, x, x^{(1)}, \dots, x^{(k)}) = 0}$$

t independent variable
x dependent variable.

A solution to an ODE is a function

$\phi \in C^k(I) \quad I \subset \mathbb{R}$

such that $\quad F(t, \phi(t), \dots, \phi^{(k)}(t)) = 0$ for $t \in I$

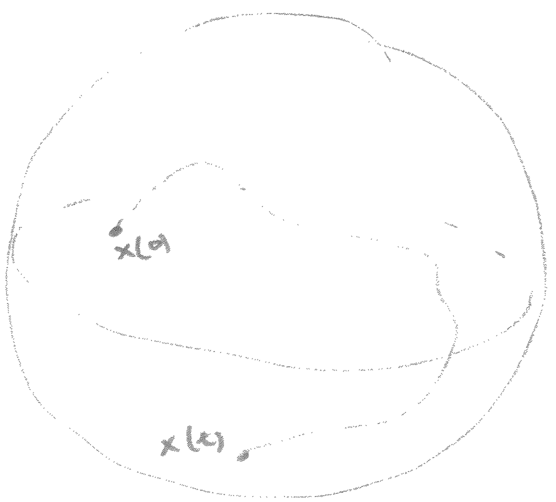
This is too general! We consider only

$$x^{(k)} = f(t, x, \dot{x}, \ddot{x}, \dots, x^{(k-1)}) \qquad x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{pmatrix}$$

$$\begin{cases} x_1^{(k)} = f_1(t, \dots, x^{(k-1)}) \\ x_m^{(k)} = f_m(t, \dots, x^{(k-1)}) \end{cases}$$

a system of ode:s of order k.

$M = S^2$



A system is

- linear if $\dot{x}_i = g_i(t) + \sum_j \sum_l f_{i,j,l}(t)\, x_l^{(i)}$

$$\left[\ \dot{x} = Ax + g\ \right]$$

- homogenous if $g_i(t) = 0$

- autonomous if t does not appear explicitly: $g_i(t) = \text{const}$, $f_{i,j,l}(t) = \text{const}$

Example

$$\ddot{x} + \sin(t)\cos(\dot{x}) = x \qquad x \in \mathbb{R}$$

Let $\begin{cases} x_1 = x \in \mathbb{R} \\ x_2 = \dot{x} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = -\sin(t)\cos(x_2) + x_1 \end{cases}$

i.e. $$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin(t)\cos(x_2) + x_1 \end{pmatrix}$$

Next, let $x_3 = t \Rightarrow \dot{x}_3 = 1$, $x_3(0) = 0$

Then $$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin(x_3)\cos(x_2) + x_1 \\ 1 \end{pmatrix}$$ autonomous

Do solutions exist?

Are they unique? No! Why?

First order equations on $\mathbb{R}$

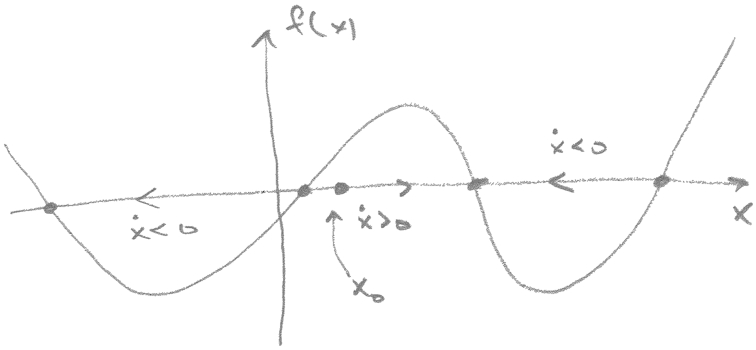
Consider

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \qquad f \in C(\mathbb{R})$$

Assume $f(x_0) \neq 0$

| what happens if $f(x_0) = 0$? |

$$\frac{dx/dt}{f(x)} = 1$$

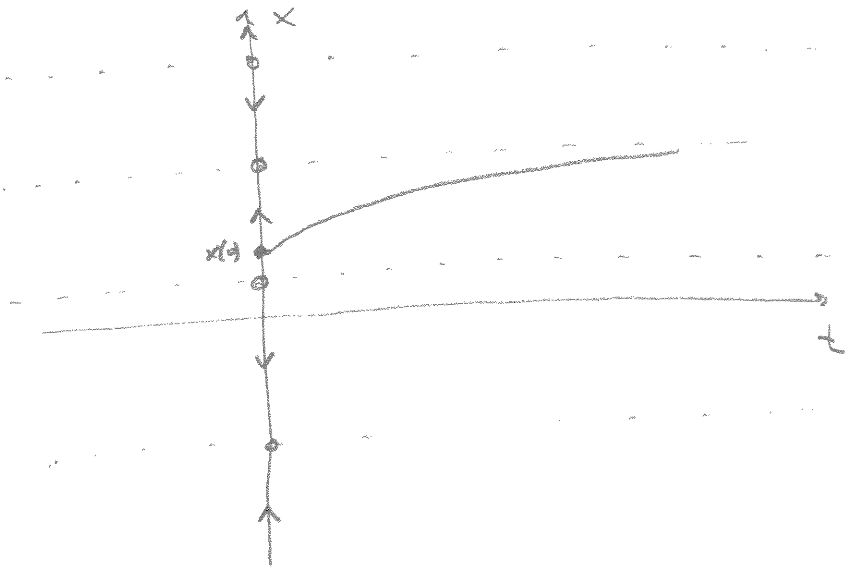


$$\int_{x_0}^{x(t)} \frac{1}{f(y)}\, dy = t = \int_0^t ds$$

Let $F(x) = \int_{x_0}^{x} \frac{1}{f(y)}\, dy$

$\Rightarrow \quad x(t) = F^{-1}(t)$

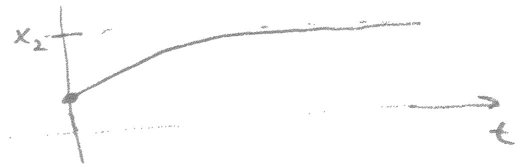
Check $F(x_0) = 0$



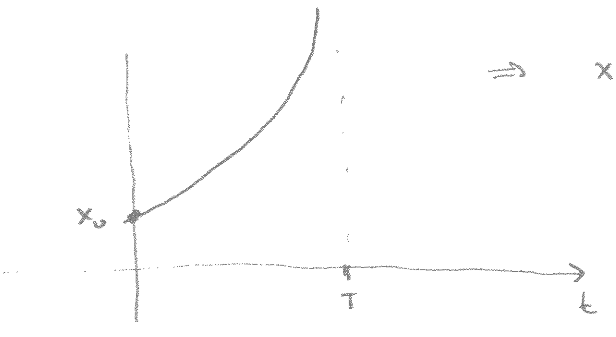
Possible cases

1) $F(x) = \int_{x_0}^{x} \frac{1}{f(y)}\, dy \to \infty$ when $x \to x_2$

i.e. $x(t) \to x_2$ when $t \to \infty$



2) $F(x) \to T$ when $x \to \infty$



$\Rightarrow \quad x(t) \to \infty$ when $t \to T$.

Examples

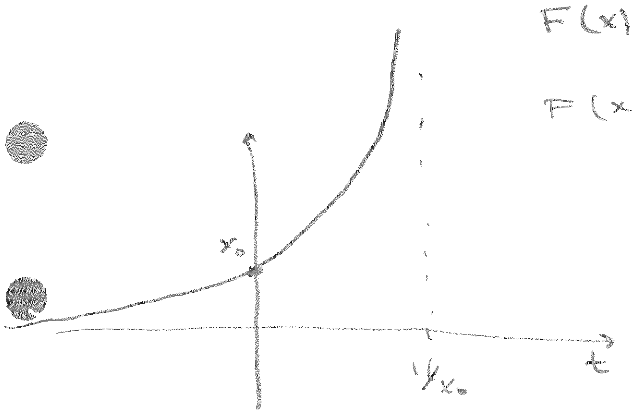
1) $f(x) = x \quad \Rightarrow \quad F(x) = \int_{x_0}^{x} \frac{1}{y}\,dy = \log \frac{x}{x_0}$

$F^{-1}(t) = x_0 e^{t}$ $\qquad \begin{cases} \dot{x} = x \\ x(0) = x_0 \end{cases}$

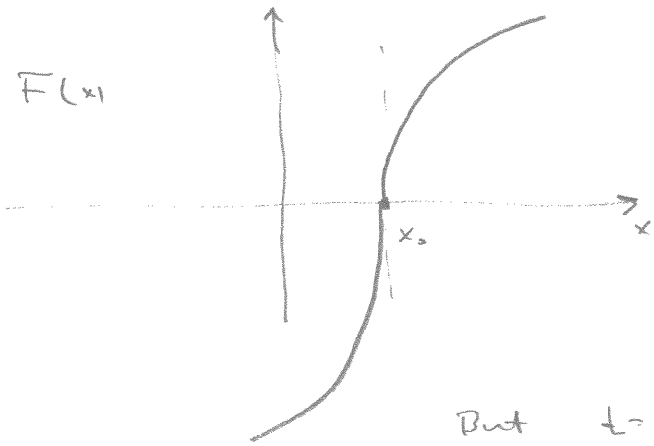
2) $f(x) = x^2 \quad \Rightarrow \quad F(x) = \int_{x_0}^{x} \frac{1}{y^2}\,dy = \frac{1}{x_0} - \frac{1}{x}$

$F(x) \to \frac{1}{x_0}$ when $x \to \infty$

$F(x) \to -\infty$ when $x \to 0$



3) $f(x) = \sqrt{|x|} \qquad F(x) = \int_{x_0}^{x} \frac{1}{\sqrt{|y|}}\,dy = \begin{cases} 2\left(\sqrt{|x_0|} - \sqrt{|x|}\right) & x_0, x < 0 \quad \cancel{x > x_0} \\ 2\left(\sqrt{|x|} - \sqrt{|x_0|}\right) & \cancel{x < x_0} \quad x_0 \cdot x > 0 \end{cases}$

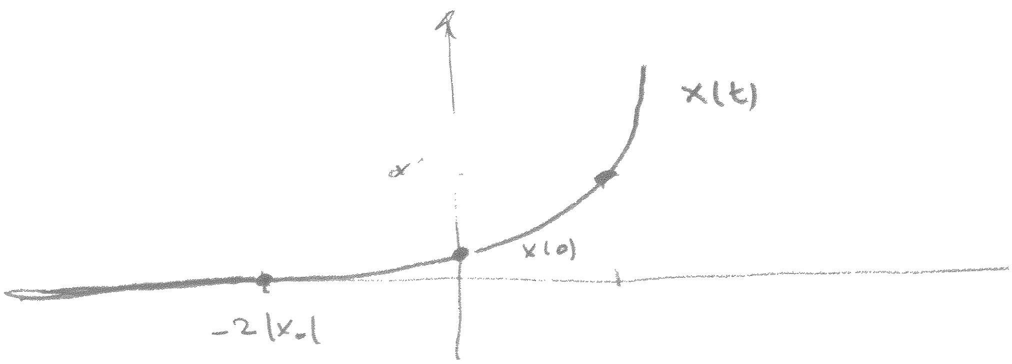


But $f(0) = 0 \Rightarrow$

$x(t) \equiv 0$ is a solution

But $t = 2\left(\sqrt{|x_0|} - \sqrt{|x|}\right) \Rightarrow x(t) = \left(\sqrt{|x_0|} + \frac{t}{2}\right)^2$

when $-2|x_0| < t < \infty$

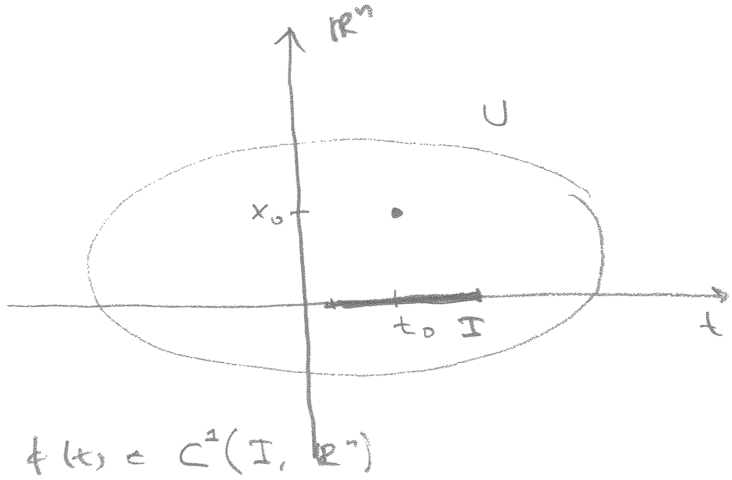


The initial value problem (IVP)

Note: A k-th order ode normally needs k "boundary conditions" for uniqueness

For a dynamical system it is most natural to specify all conditions at some initial time $t=0$

The basic existence and uniqueness result



Let $f \in C(U, \mathbb{R}^n)$

Consider

$$(*) \quad \begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

A solution: a function $\phi(t) \in C^1(I, \mathbb{R}^n)$ such that $\dot{\phi}(t) = f(t, \phi(t))$ for all $t \in I$

First: $(*)$ is equivalent to

$$(**) \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \quad , \text{ an integral equation.}$$

Why? Because if $*$ is true, $(**)$ can be obtained by integration.

And if $(**)$ holds then

- $x(t)$ is continuous

$\Rightarrow x(t) \in C^1$, so we can differentiate $(**)$ and get $(*)$

Why is this better?

Proof of existence and uniqueness

We want to show that there is a unique function $x(t)$ such that $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$

$\boxed{\text{Can we hope for that?}}$

we will need some more specifications on f!

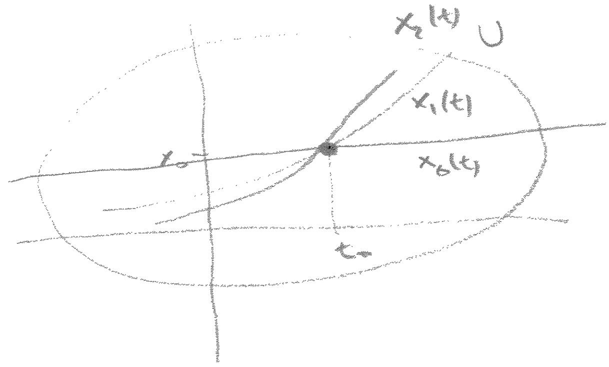
The proof is carried out by iteration:

Let $x_0(t) = x_0$ | so $x_0(t)$ is a constant function

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds$$

$\vdots$

$$x_k(t) = \underbrace{x_0 + \int_{t_0}^{t} f(s, x_{k-1}(s))\,ds}_{K[x_{k-1}](t)}$$



So $K : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$

↑ or a subset of $\mathbb{R}^n$.

An abstract formulation of the integral equation:

we want to solve $x = K[x]$

where $x \in C(\mathbb{R}^n)$.

How do you usually try to solve an equation of the form $y = G(y)$?