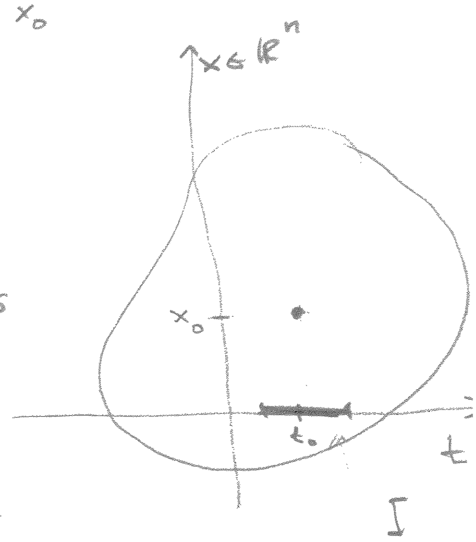
Last time:

Reformulation of the IVP $\begin{cases} \dot{x} = f(t,x) & x \in \mathbb{R}^n \\ x|_{t=t_0} = x_0 \end{cases}$

as an integral equation:



Find $x(t) \in C[I, \mathbb{R}^n)$, $x_0 \in I$,

such that

$$(*) \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

for all $t \in I$

Idea: if we define $K[x]$ as an "operator" that takes a function $x(t)$ as an argument and returns another function $y(t)$:

$$K[x](t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds .$$

Note: if $x \in C(I, \mathbb{R}^n)$, then $K[x] \in C(I, \mathbb{R}^n)$.

Then the eq. $(*)$ can be formulated as a "fixed point problem" for K: Find $x \in C[I, \mathbb{R}^n)$ such that $x = K[x]$

Try fixed point iteration:

Let $x^0(t) = x_0 \qquad (t \in I)$

and then

$$x^k(t) = x_0 + \int_{t_0}^{t} f(s, x^{k-1}(s))\, ds = K[x^{k-1}](t)$$

When does this converge?

When is there only one solution to $K[x] = x$?

Some functional analysis:

Let X be a real vector space, and $\|\cdot\|$ a norm on X

Ex $X = \mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$

Ex For a closed interval I let

$X = C(I, \mathbb{R})$, $\|x\| = \sup_{t \in I} |x(t)|$ (why sup and not max?)

[check that you remember the properties of real vectorspaces and norms]

For example: the triangle inequality must hold:

$\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in X$.

Def A sequence $\{x_n\}_{n=1}^{\infty}$ $x_n \in X$ is convergent if there is $x \in X$ such that
$\|x - x_n\| \to 0$ when $n \to \infty$.

Def A sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for each $\varepsilon > 0$ there is $N_\varepsilon > 0$ such that if $n > N_\varepsilon$ and $m > N_\varepsilon$ then $\|x_m - x_n\| < \varepsilon$.

Def A normed space such that all Cauchy sequences are convergent is called a Banach space.

Ex $C(I, \mathbb{R})$ is Banach if I is closed, bounded but not e.g. if $I = [0, \infty[$.

Def Let X be a Banach space. and let $K : C \subset X \to X$
If $x \in C$ and $K(x) = x$, x is called a fixed point
If for all $x, y \in C$, $\|K(x) - K(y)\| \le \theta \|x - y\|$, where $\theta < 1$, then K is called a contraction

Theorem **The contraction mapping principle**

Let C be closed, $C \subset X$, and assume that $K : C \to C$ is a contraction. Then K has a unique fixed point $\bar{x} \in C$, and

(*) $\| K^n(x) - \bar{x} \| \le \frac{\theta^n}{1-\theta} \| K(x) - x \| \qquad x \in C$

| what does this mean? |

Proof i) If $x = K(x)$, $y = K(y)$ then

$$\| x-y \| = \| K(x) - K(y) \| \le \theta \| x-y \| \Rightarrow (1-\theta) \| x-y \| \le 0$$

$$\Rightarrow \| x-y \| = 0 \Rightarrow x = y$$

why? , This is the uniqueness result.

ii) Now take $x_0 \in C$ and let $x_n = K^n(x_0)$

Then $\| x_{n+1} - x_n \| = \| K(x_n) - K(x_{n-1}) \| \le \theta \| x_n - x_{n-1} \| \le \theta^n \| x_1 - x_0 \|$

why?

Hence (if $n > m$)

$$\| x_n - x_m \| = \sum_{j=m+1}^{n} (x_j - x_{j-1}) \le \sum_{j=m+1}^{n} \| x_j - x_{j-1} \| \le$$

$$\le \sum_{j=m+1}^{n} \theta^j \| x_1 - x_0 \| \le \frac{\theta^m}{1-\theta} \| K(x_0) - x_0 \|$$

$\Rightarrow$ $\{x_n\}$ is a Cauchy sequence [why?]

$x_n \to \bar{x} \in C$, and [why do we do this?]

$$\| K(\bar{x}) - \bar{x} \| = \| K(\bar{x}) - K(x_n) + K(x_n) - x_n + x_n - \bar{x} \|$$

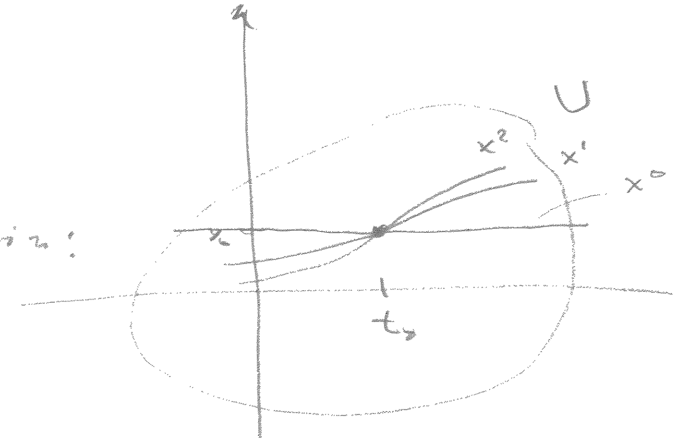
$$\le \underbrace{\| K(\bar{x}) - K(x_n) \|}_{\to 0} + \underbrace{\| K(x_n) - x_n \|}_{\to 0} + \underbrace{\| x_n - \bar{x} \|}_{\to 0} \to 0$$

when $n \to \infty$

why

$\boxed{\text{Try to prove (*)}}$

Back to the ODE



We need one more definition:

Def Let $f \in C(U, \mathbb{R}^n)$

Assume that for all $V \subset U$, V compact,

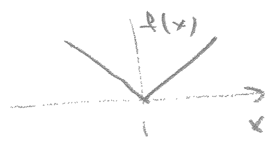
there is a constant L $(= L_V)$

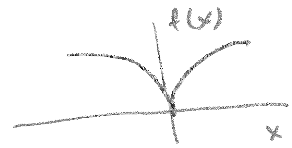
such that

$$L = \sup_{\substack{(t,x) \neq (t,y) \\ \in V \quad \in V}} \frac{|f(t,x) - f(t,y)|}{|x-y|} < \infty$$

we say that f satisfies a local Lipschitz condition

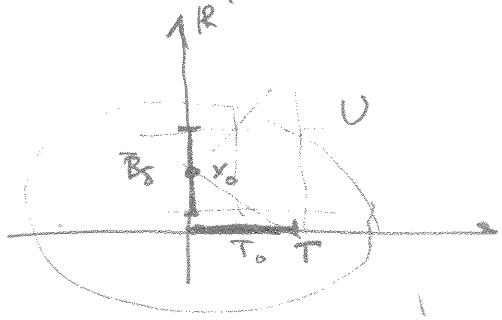
Ex $f(x) = |x|$ is Lipschitz, but not $f(x) = \sqrt{|x|}$





Now consider $\begin{cases} \dot{x} = f(t,x) \\ x|_{t_0} = x_0 \end{cases}$. We may assume $t_0 = 0$

and we consider only $t > 0$



Step I

Find $C \subset X$

$\overline{B}_\delta = \{ x \in \mathbb{R}^n : |x - x_0| \leq \delta \}$,

and let $V = [0,T] \times \overline{B}_\delta(x_0)$

Let $M = \max_{(t,x) \in V} |f(t,x)|$.

Then $|K(x)(t) - x_0| \leq \int_0^t |f(s,x(s))| \, ds \leq tM$. Let $T_0 = \min(T, \frac{\delta}{M})$

why?

and $V_0 = [0, T_0] \times \overline{B}_\delta(x_0)$

Then: Let $X = C([0,T_0], \mathbb{R}^n)$, $C = \{ x \in X : \|x - x_0\| \leq \delta \}$

Next step is to prove that K is a contraction on C. We have

$$|K(x)(t) - K(y)(t)| \le \int_0^t |f(s, x(s)) - f(s, y(s))|\, ds$$

$$\le L \int_0^t |x(s) - y(s)|\, ds \le Lt \sup_{0\le s\le t} |x(s) - y(s)|$$

$$\le LT_0 \sup_{0\le s\le T_0} |x(s) - y(s)| \quad \text{for all } t \le T_0.$$

So $\quad \| K(x) - K(y) \| \le LT_0 \| x - y\|$.

| What do we do now? ... |

Theorem Let $f \in C(U, \mathbb{R}^n)$, $U \subset \mathbb{R} \times \mathbb{R}^n$ and assume that f satisfies a local Lipschitz condition in U. Let $(t_0, x_0) \in U$ and consider

$$(*) \begin{cases} \dot{x} = f(t, x) \\ x|_{t_0} = x_0 \end{cases}$$

There is an interval $I \ni t_0$ such that $(*)$ has a unique solution $x \in C^1(I, \mathbb{R}^n)$

A more detailed calculation:

$$| K^n(x) - K^n(y) | \le \int_0^t |f(s, K(x)(s)) - f(s, K(y)(s))|\, ds$$

$$\le \int_0^t L\, |K^{n-1}(x)(s) - K^{n-1}(y)(s)|\, ds \le$$

$\underline{n=1}$ $\quad | K(x)(t) - K(y)(t) | \le \int_0^t L\, | x(s) - y(s) |\, ds \le L \sup_s |x(s) - y(s)|\, t$

$$| K^2(x)(t) - K^2(y)(t) | \le \int_0^t L\, \underbrace{|K(x)(s) - K(y)(s)|}_{\le L \sup | \, | s}\, ds$$

$$\le \sup|x(s) - y(s)|\, L^2 \int_0^t s\, ds = \sup|x(s) - y(s)|\, \frac{L^2 t^2}{2}$$

$$\cdots \to | K^n(x)(t) - K^n(y)(t) | \le \sup|x(s) - y(s)|\, \frac{L^n t^n}{n!}$$

| what good is this? |