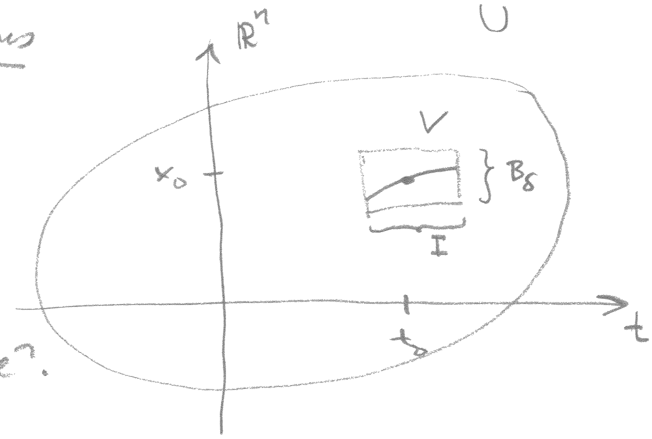
Note: If f is differentiable, you can prove that the solution is differentiable w.r.t x_0 and many other things. See the book for some of these results.

Extension of solutions

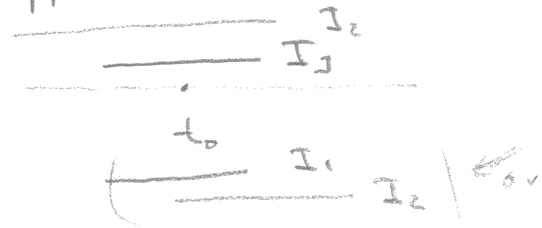
The existence proof only holds inside V, but does the solution end there?



Consider $\begin{cases} \dot{x} = f(t,x) \\ x|_{t_0} = x_0 \end{cases}$ and suppose there are two solutions,

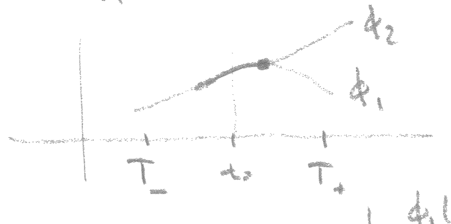
$\phi_1(t)$ for $t \in I_1$

$\phi_2(t)$ for $t \in I_2$



Let $I = I_1 \cap I_2 =]T_-, T_+[$ ($\ni t_0$, so $I \neq \emptyset$)

Then $\phi_1(t) = \phi_2(t)$ for all $t \in I$, because by uniqueness we can't have



Now define $\phi(t) = \begin{cases} \phi_1(t) & t \in I_1 \\ \phi_2(t) & t \in I_2 \end{cases}$ (no ambiguity!)

which is now a well defined solution on $I_1 \cup I_2$

But when can it happen that a solution cannot be extended?

Lemma Let $\phi(t)$ solve $\begin{cases} \dot{x} = f(t,x) \\ x|_{t_0} = x_0 \end{cases}$

in $I =]t_-, t_+[\ni t_0$.

There is $\varepsilon > 0$ such that the solution can be extended to $]t_-, t_+ + \varepsilon[$ if and only if

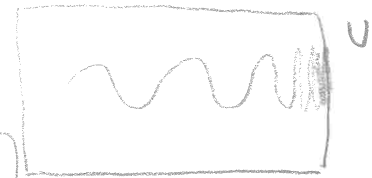
$$\lim_{t \nearrow t_+} (t, \phi(t)) = (t_+, y) \in U \quad \text{exists.}$$

Examples

1) $|\phi(t)| \to \infty$ when $t \nearrow t_+$

2) $(t, \phi(t)) \to (\bar{t}, \bar{x}) \in \partial U$

3)



$$\boxed{\dot{x} = \frac{-1}{2x} \quad t > 0 \quad x > 0}$$

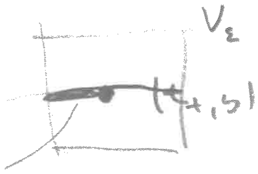
$$x(t) = \sqrt{x_0^2 - t}$$

$$\boxed{\ddot{x} + \frac{2}{t}\dot{x} + \frac{1}{t^4}x = 0}$$

$$\left(x(t) = \sin \frac{1}{t}\right)$$

Proof of lemma If the extension exists, the limit must exist. [why?]

If the limit $(t_+, y) \in U$ exists there is an interval $(t_+ - \varepsilon, t_+ + \varepsilon)$ such that $\begin{cases} \dot{x} = f(t,x) \\ x(t_+) = y \end{cases}$ has a solution there



by uniqueness the two must coincide here.

Equations with linear growth

<u>Theorem</u> Let $U = \mathbb{R} \times \mathbb{R}^n$, $|f(t,x)| \le M(T) + L(T)|x|$ $t \in [-T,T]$

and assume that this holds for every $T > 0$.

Then the solution to $\begin{cases} \dot{x} = f(t,x) \\ x(0) = x_0 \end{cases}$

exists for all $t < \infty$.

<u>Lemma</u> Let $\phi(t)$ be the solution to

$$\begin{cases} \dot{x} = f(t,x) & t \in \,]t_-, t_+[\\ x(t_0) = x_0 \end{cases}$$

Assume that $(t, \phi(t)) \in [t_-, t_+] \times C \subset U$
where C is closed and bounded.
Then there is $\varepsilon > 0$ so that $\phi(t)$ can be extended
to $]t_-, t_+ + \varepsilon[$.

| what could be the problem? |

<u>Proof</u> Let $t_n \to t_+$. Then $|\phi(t_n) - \phi(t_m)|$

$$\le \int_{t_m}^{t_n} |f(s, \phi(s))| ds \le M |t_n - t_m|$$, so $\phi(t_n)$ is Cauchy

what is M?

$\Rightarrow \phi(t_n) \to \phi_+$ and $(t_n, \phi(t_n)) \to (t_+, \phi_+)$.

Then solve $\begin{cases} \dot{x} = f(t,x) \\ x(t_+) = \phi_+ \end{cases}$.

Proof of theorem

Let ϕ solve $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases} \Rightarrow \phi(t) = x_0 + \int_{t_0}^{t} f(s,\phi(s))\,ds$

$$\Rightarrow \quad |\phi(t)| \le |x_0| + \int_0^t \left(M(T) + L(T)|\phi(s)|\right) ds$$

$$\Rightarrow \ldots \quad |\phi(t)| \le \underbrace{|x_0| e^{L(T)T} + \frac{M(T)}{L(T)}\left(e^{L(T)T} - 1\right)}$$

bounded = $(t, \phi(t))$ belongs to a compact set C, so the Lemma implies that $\phi(t)$ can be extended beyond $t = T$.

Numerical methods

Many different. Some explained in computer exercise. Easiest: Euler.

$\begin{cases} \dot{x} = f(t,x) \\ x(0) = x_0 \end{cases} \rightsquigarrow$ consider $t_0 = 0 < t_1 < t_2 < t_3 \ldots$
If $t_1 - t_0$ is small, we should have

$$x(t) = x_0 + \int_0^{t_1} f(s, x(s))\,ds$$

$$\approx x_0 + \int_0^{t_1} f(0, x(0))\,ds =$$

$$= x_0 + (t_1 - t_0) f(0, x_0) \equiv x_1$$

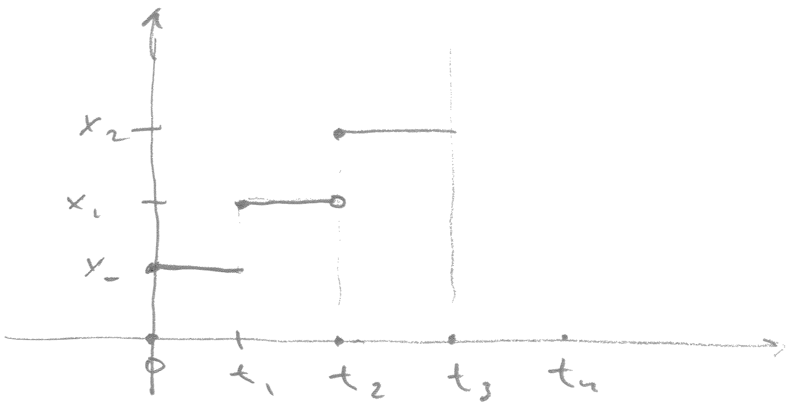
Then continue:

$$x_k = x_{k-1} + (t_k - t_{k-1}) f(t_{k-1}, x_{k-1})$$

This is the explicit Euler method.

Is this a good method? Does it converge?

Let $h = \max |t_k - t_{k-1}|$



Let $x^h(t) = x_k$ when $t_{k-1} \le t < t_k$

Question: Is it true that $\sup_{0 \le t \le T} |x^h(t) - x(t)| \to 0$?

How fast?

Linear systems

$$\begin{cases} \dot{x}(t) = a\,x(t) \\ x(0) = x_0 \end{cases} \qquad x(t) \in \mathbb{R}$$

$$\Rightarrow \quad \boxed{x(t) = e^{at} x_0} = \exp(at)\, x_0$$

in $\mathbb{R}^n$:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n & x_1(0) = x_{10} \\ \vdots & \vdots \\ \dot{x}_n = a_{n1}x_1 + \qquad + a_{nn}x_n & x_n(0) = x_{n0} \end{cases}$$

In matrix form:

$$(*) \quad \begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \qquad A = \begin{pmatrix} a_{11} & \dots a_{1n} \\ a_{n1} & \dots a_{nn} \end{pmatrix}$$

$(*)$ satisfies all conditions for existence and uniqueness.

$\boxed{\text{Check that!}}$

$\Rightarrow \quad x(t) = e^{tA} x_0$ We only need to define e^{tA} in a good way.

In two dimensions:

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

i) assume A can be diagonalized:

$$U^{-1}AU = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \qquad U \text{ invertible.}$$

Let $x = Uy \qquad \dot{x} = U\dot{y}$ (why?)

$$\Rightarrow \quad U^{-1}U\dot{y} = U^{-1}AU\dot{y} \iff \dot{y} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} y$$

$$y(0) = U^{-1}x_0 = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix}$$

Then $y_1(t) = e^{t\alpha_1} y_{01}$

$y_2(t) = e^{t\alpha_2} y_{02}$ and

$$x(t) = U \begin{pmatrix} e^{t\alpha_1} & 0 \\ 0 & e^{t\alpha_2} \end{pmatrix} U^{-1} x_0$$

Then $x_i(t)$ is a linear combination of $e^{t\alpha_1}$, $e^{t\alpha_2}$

If A is not diagonalizable, there is U such that

$$U^{-1}AU = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \Rightarrow \qquad (x = Uy)$$

$$\dot{y} = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} y \qquad \Rightarrow \qquad y_2(t) = e^{\alpha t} y_{01}$$

$$\dot{y}_1 = \alpha y_1 + y_2 = \alpha y_1 + e^{\alpha t} y_{01}$$

$$\dot{y}_1 - \alpha y_1 = e^{\alpha t} y_{02}$$

$$\frac{d}{dt}(e^{-\alpha t} y_1) = y_{01}$$

$$\Rightarrow e^{-\alpha t} y_1 = y_1(0) + t\, y_{02}$$

$$\Rightarrow y_1(t) = e^{\alpha t} \underbrace{(y_{10} + t y_{02})}_{\text{polynomial}}$$

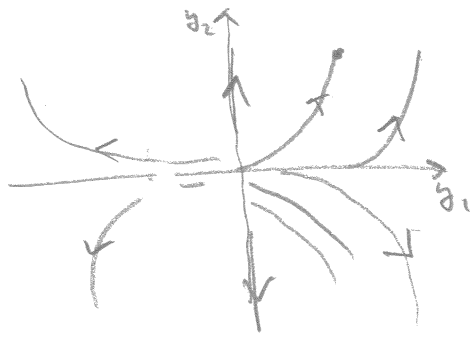
If $U^{-1}AU = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ with

$\alpha_1 = \alpha_2^* = \lambda + i\omega$, $\quad x(t)$ linear comb of $e^{\lambda t}\cos\omega t$

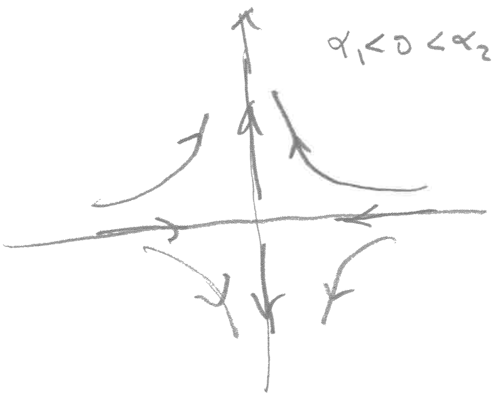
and $e^{\lambda t}\sin\omega t$

$\dot{y} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} (y)$

$0 < \alpha_1 < \alpha_2$

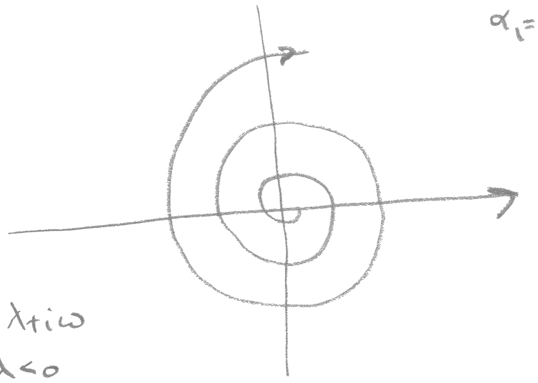


$\alpha_1 < 0 < \alpha_2$



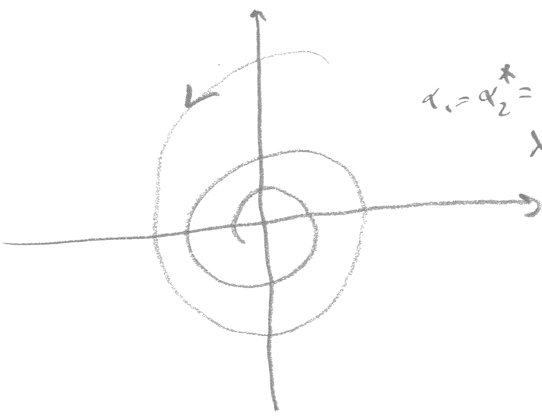
$\alpha_1 = \alpha_2^* = \lambda + i\omega$

$\lambda > 0$

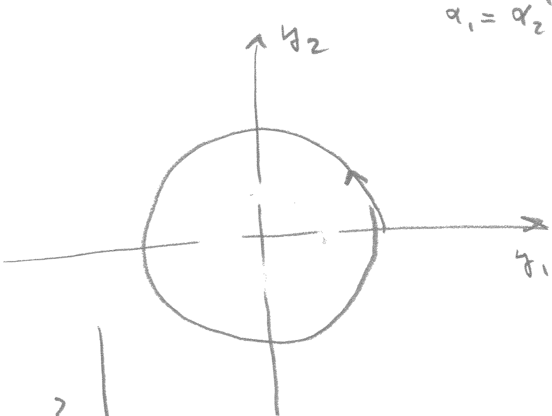


$\alpha_1 = \alpha_2^* = \lambda + i\omega$

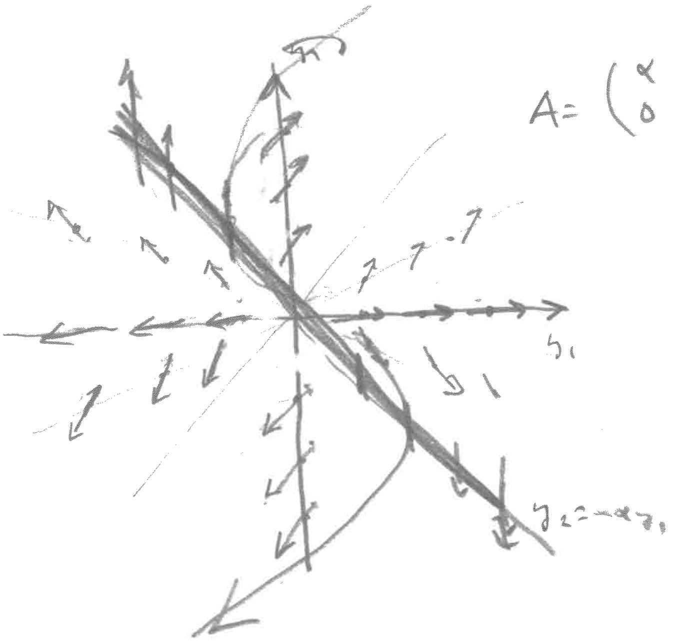
$\lambda < 0$



$\alpha_1 = \alpha_2^* = i\omega$



| If $y(t)$ is like this, what about $x(t) = U y(t)$? |



$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ $\quad \alpha > 0$

$\dot{y}_1 = \alpha y_1 + y_2$

$\dot{y}_2 = \alpha y_2$

$\Rightarrow \dfrac{dy_2}{dy_1} = \dfrac{\alpha y_2}{\alpha y_1 + y_2}$

$|y_2| \ll \alpha |y_1| \Rightarrow \dfrac{dy_2}{dy_1} \approx \dfrac{y_2}{y_1}$

$\alpha y_1 + y_2 = 0$

$\Rightarrow y_2 = -\alpha y_1$

Def A linear system is called stable
if all solutions remain bounded as $t \to \infty$
and asymptotically stable if
all solutions $x(t) \to 0$ when $t \to \infty$.