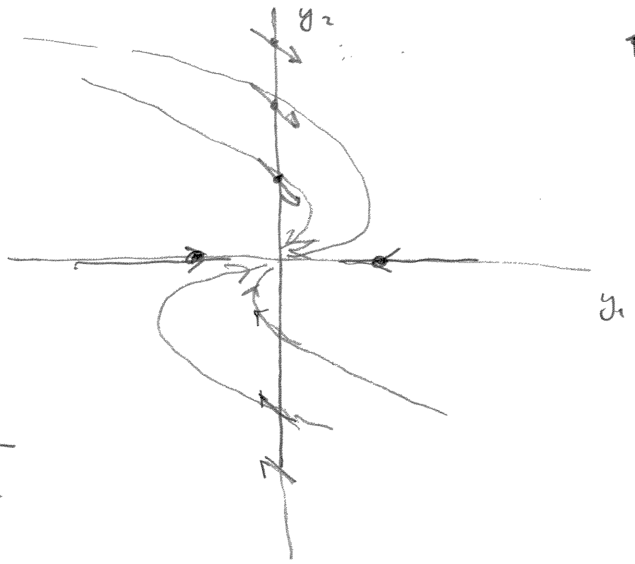
$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector ... the only one.

$\frac{dy}{dt} = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ If $y_0 = \begin{pmatrix} y_{01} \\ 0 \end{pmatrix}$, then

$$y(t) = \begin{pmatrix} e^{\alpha t} y_{01} \\ 0 \end{pmatrix}$$



~~To~~ $\alpha < 0$

$$\frac{dy_1}{dt} = \alpha y_1 + y_2$$

$$\frac{dy_2}{dt} = \alpha y_2$$

$$\frac{dy_2}{dy_1} = \frac{\alpha y_2}{\alpha y_1 + y_2}$$

The matrix exponential

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k}{k!} t^k \qquad \text{when } a \in \mathbb{C}$$

convergent for $t \in \mathbb{R}$.

This works also for matrices: A matrix

$$A = \begin{pmatrix} a_{11} & \cdot & a_{1n} \\ a_{n1} & & a_{nn} \end{pmatrix} \qquad a_{jk} \in \mathbb{R} \quad (\text{or } \mathbb{C}, \text{ works the same})$$

Let $A^k = \underbrace{A \cdots A}_{k \text{ times}}$ and $A^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & \ddots \; 1 \end{pmatrix}$

Formally

$$e^{tA} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k$$

And (also formally)

$$\frac{d}{dt}\left(e^{tA}\right) = \sum_{k=1}^{\infty} \frac{A^k}{k!} k t^{k-1} = A \sum_{k=1}^{\infty} \frac{A^{k-1}}{(k-1)!} t^{k-1} = A \sum_{k=0} \frac{A^k}{k!} t^k$$

Hence: if $x(t) = e^{tA} x_0$

$$\begin{cases} \frac{d}{dt} x(t) = A e^{tA} x_0 = A\, x(t) \\ x(0) = x_0 . \end{cases}$$

This can be seen also by Picard iteration:

$$x_0(t) = x_0$$

$$x_1(t) = x_0 + \int_0^t A x_0(s) ds = x_0 + tA x_0$$

$$x_2(t) = \ldots \; x_0 + tAx_0 + \frac{t^2}{2} A^2 x_0 \; \ldots\ldots$$

The <u>Matrix norm</u> is defined as

$$\| A \| = \sup_{x,\, |x|=1} |Ax| = \sup_{x \neq 0} \frac{|Ax|}{|x|}$$

why are these equal?

is it necessary to take sup instead of max?

The linear space of $n \times n$ matrices is a Banach space

Also $\| A^j \| \leq \|A\|^j$ (is this obvious?)

and $\|A\|$ is a <u>norm</u> (why)

Therefore : $\left\| \sum_{j=0}^{n} \frac{1}{j!} A^j \right\| \leq \sum_{j=0}^{n} \frac{\|A\|^j}{j!} \Rightarrow$

Series is norm convergent.

But while $e^{(a+b)} = e^a e^b$ when $a, b \in \mathbb{C}$,

this is not true when A, B are matrices

unless $AB = BA$, i.e if $[A,B] \equiv AB - BA = 0$

<u>Lemma</u> if $[A,B]=0$, then $\exp(A+B) = \exp(A)\exp(B)$
$= \exp(B)\exp(A)$.

<u>Lemma</u> if U is invertible,

$$U^{-1} \exp(A)\, U = \exp(U^{-1} A U)$$

<u>Theorem</u> The Jordan canonical form

Let A be an $n \times n$ -matrix. There is an invertible U such that

$$U^{-1}AU = \begin{pmatrix} J_1 & & 0 \\ 0 & \ddots & \\ 0 & & J_m \end{pmatrix}$$

where

$$J_k = \alpha I + N = \begin{pmatrix} \alpha_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha_k \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

<u>Proof</u> in appendix. not part of the core.

<u>Homework</u>: N is called "nilpotent" because there is $k \geq 1$ such that $N^k = 0$.

[prove that].

Why is this important?

1) Consider an equation of the form

$$\dot{x} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) x$$

where A is $m \times m$ and B is $k \times k$ with $m+k=$

Then $\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$

and $\frac{d}{dt}\begin{pmatrix} x_{m+1} \\ \vdots \\ x_n \end{pmatrix} = B \begin{pmatrix} x_{m+1} \\ \vdots \\ x_n \end{pmatrix}$

So these are two decoupled systems of odes.

In term of the matrix exponential.

$$\left(\begin{array}{c|c} A^k & 0 \\ 0 & B^k \end{array}\right)\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = \left(\begin{array}{c|c} A^{k+1} & 0 \\ \hline 0 & B^{k+1} \end{array}\right)$$

So

$$\exp\left(\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)\right) = \left(\begin{array}{c|c} \exp A & 0 \\ \hline 0 & \exp B \end{array}\right)$$

And ... it is easy to compute

$$\exp\begin{pmatrix} \alpha & 1 & 0 \\ & \ddots & 1 \\ 0 & & \alpha \end{pmatrix} \quad \text{because}$$

$$\begin{aligned}
(\alpha I + N)^0 &= I \\
(\alpha I + N)^1 &= \alpha I + N \\
(\alpha I + N)^2 &= \alpha^2 I + 2\alpha N + N^2 \\
(\alpha I + N)^j &= \sum_{k=0}^{j} \binom{j}{k} \alpha^{k-j} I N^k
\end{aligned}$$

4×4:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad N^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad N^4 = 0$$

$$\begin{aligned}
\Rightarrow \quad \exp(\alpha I + N) &= \exp(\alpha I) \exp N \\
&= e^\alpha \left(I + N + \tfrac{1}{2} N^2 + \dots + \tfrac{1}{(k-1)!} N^{k-1} \right) \\
&= e^\alpha \begin{pmatrix} 1 & 1 & \frac{1}{2} & \\ & \ddots & \ddots & \frac{1}{2} \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}
\end{aligned}$$

and more importantly:

$$\exp\left(t\left(\alpha I + N\right)\right) =$$

$$= e^{\alpha t}\left(I + tN + \frac{t^2}{2}N^2 + \dots + \frac{t^{k-1}}{(k-1)!}N^{k-1}\right)$$

$$= e^{\alpha t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & & & t^2/2 \\ & 0 & & & t \\ & & & & 1 \end{pmatrix}$$

So the solution of $\dot{x} = Jx$ is an exponential $e^{\alpha t}$ times a polynomial of degree $\leq k-1$ if k is the size of the Jordan block.

<u>Theorem</u> A solution to $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ converges to 0 if and only if x_0 lies in the subspace spanned by the generalized eigenspaces corresponding to eigenvalues α with $\operatorname{Re}\alpha < 0$ and remain bounded if x_0 lies in the generalized subspace corresponding to eigenvalues with negative real part and <u>eigenspaces</u> with zero real part.

<u>Proof</u>

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \Leftrightarrow$$

$x = Uy$

$U^{-1}AU$ jordan

$$\begin{cases} \dot{y} = Jy \\ y(0) = U^{-1}x_0 \end{cases} \qquad J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_s \end{pmatrix}$$

If $J_s = \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{pmatrix}$ $\begin{cases} \dot{y}_s = J_s y \\ y_s(0) = y_{s0} \end{cases}$ has solution $e^{\alpha t} y_{s0}$

So: $\operatorname{Re} \alpha < 0 \Rightarrow y_s(t) \to 0$

$\operatorname{Re} \alpha = 0 \Rightarrow y_s(t)$ bounded

$\operatorname{Re} \alpha > 0 \Rightarrow |y_s(t)| \to \infty$.

If $J_s = \begin{pmatrix} \alpha & 1 & 0 \\ & \ddots & 1 \\ 0 & & \alpha \end{pmatrix}$ then

$y_s(t) = e^{\alpha t} \cdot$ (polynomial of degree $\leq k-1$)

$\Rightarrow y_s(t) \to 0$ if $\operatorname{Re}\alpha < 0$

$|y_s(t)| \to \infty$ if $\operatorname{Re}\alpha \geq 0$.

<u>Def</u> A linear system is <u>stable</u> if all solution remain bounded as $t \to \infty$

and <u>asymptotically stable</u> if all solutions converge to zero when $t \to \infty$.

Invariant subspaces (a small detour)

The Cayley Hamilton theorem

Let A be an $n \times n$ - matrix (real or complex)
and let $p_A(\lambda) = \det(\lambda I - A)$. Then $p_A(A) = 0$

[i.e. if $p_A(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + p_0$
then $A^n - c_{n-1} A^{n-1} \dots p_0 I = 0$]

Proof Write
$$I = (\lambda I - A)(\lambda I - A)^{-1}$$ [when is this ok?]

Cramer's rule: $B^{-1} = \frac{1}{\det B}\begin{bmatrix} \tilde{b}_{11} & \dots & \tilde{b}_{1n} \\ \vdots & & \\ & & \tilde{b}_{nn} \end{bmatrix}$

where $\tilde{b}_{nn}$ are subdeterminants of B.

$$\Rightarrow (\lambda I - A)^{-1} = \frac{1}{p_A(\lambda)} \begin{pmatrix} p_{11}(\lambda) & \dots & p_{1n}(\lambda) \\ & & \\ p_{n1}(\lambda) & \dots & p_{nn}(\lambda) \end{pmatrix}$$

$p_{ij}(\lambda)$ polynomials of deg. $\leq n-1$.

$$\Rightarrow p_A(\lambda)(\lambda I - A)^{-1} = \lambda^{n-1} B_{n-1} + \dots + \lambda B_1 + B_0$$

$$\Rightarrow p_A(\lambda) I = (\lambda I - A)(\lambda^{n-1} B_{n-1} + \dots + \lambda B_1 + B_0)$$
$$= \lambda^n B_{n-1} + \lambda^{n-1} B_{n-2} + \dots + \lambda^2 B_1 + \lambda B_0$$
$$- \lambda^{n-1}(A B_{n-1}) - \dots - \lambda^2 A B_2 - \lambda A B_1 - A B_0$$

$I = B_{n-1}$

$c_{n-1} I = B_{n-2} - A B_{n-1}$ $\quad c_2 I = B_1 - A B_2$ $\quad c_1 I = B_0 - A B_1$

$c_0 = A B_0$

$$p_A(A) I = \cancel{A^n} + A^{n-1}(B_{n-2} - \cancel{A B_{n-1}}) + \dots + A(B_0 - A B_1) - \cancel{A B_0} = 0.$$

Invariant subspaces

Let V be a vector space and $A : V \to V$.
Let V_1 be a linear subspace of V: $V_1 \subset V$.

V_1 is said to be invariant under A if
for all $v \in V_1$, $Av \in V_1$.

Ex $\mathcal{N}(A) = \{ v \in V : Av = 0 \}$

Ex If $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, then $V_1 = \left\{ v : v = \begin{pmatrix} v_1 \\ \vdots \\ v_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$ is
invariant.

Ex if $A : \mathbb{C}^n \to \mathbb{C}^n$ then
$\mathcal{N}(p(A)) = \mathbb{C}^n$ (we identify an operator with its matrix in some basis)

Thm If $p_A(\lambda) = p_1(\lambda)\, p_2(\lambda)$ where $p_1(\lambda)$ and $p_2(\lambda)$ are polynomials without common factors, then
$\mathcal{N}(p_1(A))$ and $\mathcal{N}(p_2(A))$ are invariant

$\mathcal{N}(p_1(A)) \cap \mathcal{N}(p_2(A)) = \{0\}$, and

$\forall v \in \mathbb{C}^n$, $v = v_1 + v_2$ with $v_1 \in \mathcal{N}(p_1(A))$
$v_2 \in \mathcal{N}(p_2(A))$.

Proof $v \in \mathcal{N}(p_i(A)) \Rightarrow$

$$p_i(A)Av = A p_i(A)v = 0$$

$\uparrow$ $\boxed{\text{Why?}}$

Next There are $q_1(\lambda)$, $q_2(\lambda)$ such that

$$p_1(\lambda)q_1(\lambda) + p_2(\lambda)q_2(\lambda) = 1$$ (Euclidean algorithm)

$$\Rightarrow \quad p_1(A)q_1(A) + p_2(A)q_2(A) = I$$

$$\Rightarrow \quad \underbrace{p_1(A)q_1(A)v}_{v_2} + \underbrace{p_2(A)q_2(A)v}_{v_1} = v$$

$$p_2(A)v_2 = \underbrace{p_2(A)p_1(A)}_{=0}q_1(A)v = 0 \Rightarrow v_2 \in \mathcal{N}(p_2(A)).$$

If $p_2(A)v = p_1(A)v = 0 \Rightarrow v = 0$

because $v = q_1(A)p_1(A)v + q_2(A)p_2(A)v = 0$.

If $v = v_1 + v_2 = w_1 + w_2$

$$\Rightarrow \underbrace{v_1 - w_1}_{\in \mathcal{N}(p_1(A))} = \underbrace{w_2 - v_2}_{\in \mathcal{N}(p_2(A))}$$

$$\Downarrow$$

$$\boxed{\begin{matrix} v_1 - w_1 = 0 \\ v_2 - w_2 = 0 \end{matrix}}$$

Let

$$V_\lambda = \{v : Av = \lambda v\} = \{v : (\lambda I - A)v = 0\}$$

$$\tilde{V}_\lambda = \{v : \exists\, k \geq 1 \text{ such that } (\lambda I - A)^k v = 0\}$$

V Eigenspace

$\tilde{V}_\lambda$ = generalized eigenspace

Lemma! $\tilde{V}_\lambda$ is invariant:

suppose that $(\lambda I - A)^k v = 0$.

Then $(\lambda I - A)^k Av = A(\lambda I - A)^k v = 0$. (this holds for all k).

The spaces $\tilde{V}_\lambda$ correspond to the Jordan blocks,

Two examples

1) $$\begin{cases} \dot{x} = Ax + g(t) \\ x(0) = x_0 \end{cases}$$

$$\Rightarrow \quad x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} g(s)\, ds$$

| Check this! |

2) Higher order autonomous linear equations (scalar)..

$$x^{(n)} + c_{n-1} x^{(n-1)} + c_{n-2} x^{(n-2)} + \dots + c_1 \dot{x} + c_0$$

Let $$\mathbb{x} = \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n-1)} \end{pmatrix} \quad \Rightarrow \quad \dot{\mathbb{x}} = \begin{pmatrix} 0 & 1 & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \\ & & & & \\ 0 & & & 0 & 1 \\ -c_0 & -c_1 & & & -c_{n-1} \end{pmatrix} \mathbb{x}$$

$$\mathbb{x}(0) = \begin{pmatrix} x(0) \\ \vdots \\ \vdots \\ x^{(n-1)}(0) \end{pmatrix}$$

| Does this give the right number of initial values? |

General first order systems

$$\circledast \quad \begin{cases} \frac{d}{dt} x = A(t) x \\ x(t_0) = x_0 \end{cases} \qquad A(t) = \begin{pmatrix} a_{11}(t) & & \\ & & \\ & & a_{nn}(t) \end{pmatrix}$$

Need to remember that in general $AB \neq BA$

Theorem Assume that $A \in C(I, \mathbb{R}^{n \times n})$

Then $\circledast$ has a unique solution defined on I

[why?]

But the solution is ***not*** $e^{-\int_0^t A(s)\,ds} x_0$ unless $n = 1$.

or if $A(s_1) A(s_2) = A(s_2) A(s_1)$ for all $s_1, s_2 \in I$.

But $\circledast$ is linear! $\dot{x} = A(t) x$ and $\dot{y} = A(t) y$

$$\Rightarrow \quad \frac{d}{dt}(x+y) = A(t)(x+y).$$

Let $\begin{cases} \dot{\phi}_j = A(t) \phi_j \\ \phi_j(t_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow \text{position } j. \end{cases}$

and let $\Pi(t, t_0) = \begin{pmatrix} | & | & & | \\ \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \\ | & | & & | \end{pmatrix}$

Then $\Pi(t, t_0)$ is called "the matrix solution, and it satisfies

$$\begin{cases} \frac{d}{dt}\Pi(t, t_0) = A(t)\,\Pi(t, t_0) \\ \Pi(t_0, t_0) = I \end{cases}$$

<u>Theorem</u> The solutions of ⊛ form an n-dimensional vectorspace, which is a subspace of $C(I, \mathbb{R}^n)$. A basis is given by $\{\phi_1, \dots \phi_n\}$.

<u>Proof</u> need to prove that the $\phi_1 \dots \phi_n$ are independent.

1) Note that $\Pi(t, t_1)\,\Pi(t_1, t_0) = \Pi(t, t_0)$

This is because

$$\frac{d}{dt}\left(\Pi(t, t_1)\,\Pi(t_1, t_0)\right) = \dot{\Pi}(t, t_1)\,\Pi(t_1, t_0) = A\,\Pi(t, t_1)\Pi(t_1, t_0)$$

and $\dot{\Pi}(t, t_0) = A\,\Pi(t, t_0)$

and $\underbrace{\Pi(t_1, t_1)}_{=I}\,\Pi(t_1, t_0) = \Pi(t_1, t_0)$

Now take $t = t_0$. Then $\Pi(t_0, t_1)\,\Pi(t_1, t_0) = I$

$\Rightarrow \Pi(t_0, t_1) = \Pi(t_1, t_0)^{-1}$

so $\Pi(t, t_0)$ is invertible $\Rightarrow (\phi_1 \dots \phi_n)$ are independent.

$\boxed{\text{why can't there be more?}}$

Def Let $\varphi_1 \dots \varphi_n$ be n solutions

solutions to $\dot{x} = A(t)x$

and let
$$U(t) = \begin{pmatrix} | & & | \\ \varphi_1(t) & \dots & \varphi_n(t) \\ | & & | \end{pmatrix}$$

The Wronskian is
$$W(t) = \det U(t).$$

If $\det U(t) \neq 0$ for all $t \in I$, $U(t)$ is a fundamental matrix

One can always compute
$$\Pi(t, t_0) = U(t) U(t_0)^{-1} \qquad \boxed{\text{why?}}$$

Lemma $\frac{dW}{dt} = \operatorname{tr} A(t) W(t) \Rightarrow W(t) = W(t_0) e^{\int_{t_0}^{t} \operatorname{tr} A(s) ds}$

(so either $W(t) \equiv 0$ or $W(t) \neq 0$ for all t)

Inhomogeneous equation

$$\begin{cases} \dot{x} = A(t)x + g(t) & A \in C(I, \mathbb{R}^{n\times n}) \\ x(t_0) = x_0 & g \in C(I, \mathbb{R}^n) \end{cases}$$

Ansatz Look for a solution of the form

$$x(t) = \Pi(t, t_0)\, c(t) \qquad \text{where } c(t_0) = x_0$$

$$\dot{x}(t) = A(t)\Pi(t,t_0)c(t) + \Pi(t,t_0)\dot{c}(t)$$

$$Ax(t) = A(t)\Pi(t,t_0)c(t)$$

$$\Rightarrow \quad A(t)\Pi(t,t_0)c(t) + \Pi(t,t_0)\dot{c}(t) =$$

$$= A(t)\Pi(t,t_0)c(t) + g(t)$$

$$\Rightarrow \quad \Pi(t,t_0)\dot{c}(t) = g(t)$$

$$\Rightarrow \quad \dot{c}(t) = \Pi(t,t_0)^{-1} g(t)$$

$$c(t) = x_0 + \int_{t_0}^{t} \Pi(t_0,s)\, g(s)\, ds$$

$$\Rightarrow \quad x(t) = \Pi(t,t_0)x_0 + \int_{t_0}^{t} \underbrace{\Pi(t,t_0)\Pi(t_0,s)}_{\Pi(t,s)}\, g(s)\, ds$$